Portfolio Optimization for an Illiquidity Asset with a Random Liquidation Time

Abstract

Working in the Merton’s optimal consumption framework, with continuous time, we consider an optimization problem for a portfolio with an illiquid, a risky and a risk-free asset. While a standard Black-Scholes market describes the liquid part of the investment the illiquid asset is sold at a random moment with prescribed liquidation time distribution. We study the exponential and Weibull distributions. The investor has the logarithmic utility function.

Economical setting of the problem

Investor’s portfolio includes a riskless bond $B_t$, a risky asset $S_t$ and a non-traded asset $H_t$ that generates stochastic income, i.e., dividends. The liquidation time of the portfolio $\tau$ is a random-distributed continuous variable. The risk-free bank account $B_t$, with the constant interest rate $r$ and the stock price $S_t$ with the continuously compounded rate of return $\alpha > r$ and the standard deviation $\sigma$ follow

\[
dB_t = rB_t dt, \quad dS_t = S_t(\alpha dt + \sigma dW_t^1), t \leq \tau. \tag{1}
\]

The illiquid asset $H_t$, that can not be traded up to the time $\tau$ and its paper value is correlated with the stock price

\[
dH_t = (\mu - \delta) dt + \eta (\rho dW_t^1 + \sqrt{1 - \rho^2} dW_t^2), t \leq \tau. \tag{2}
\]

where $\mu$ is the expected rate of return of the risky illiquid asset, $(W^1, W^2)$ are two independent standard Brownian motions, $\delta$ is the rate of dividend paid by the illiquid asset, $\eta$ is the continuous standard deviation of the rate of return, and $\rho \in (-1, 1)$ is the correlation coefficient between the stock index and the illiquid risky asset. The parameters $\mu, \delta, \eta, \rho$ are all assumed to be constant.

The stochastically distributed time $\tau$ does not depend on the Brownian motions $(W^1, W^2)$. The probability density function of the $\tau$ distribution is denoted by $\phi(t)$, whereas $\Phi(t)$ denotes the cumulative distribution function, and $\overline{\Phi}(t)$, the survival function, also known as a reliability function, $\overline{\Phi}(t) = 1 - \Phi(t)$.

Investor has an allocation-consumption plan $(\pi, c)$ respectively. The consumption stream $c(t)$ is admissible if and only if it is positive and there exists a strategy that finances it. The liquid wealth process $L_t$ must cover the consumption stream.

The wealth process $L_t$ is the sum of cash holdings in bonds, stocks and random dividends from the non-traded asset minus the consumption stream, i.e. it must satisfy the balance equation

\[
dL_t = (rL_t + \delta H_t + \pi(\alpha - r) - c_t) dt + \pi \sigma dW_t^1. \tag{3}
\]

The investor wants to maximize the overall utility consumed up to the random time of liquidation $\tau$, given by

\[
U(c) := E \left[ \int_0^\infty \overline{\Phi}(t) U(c(t)) dt \right]. \tag{4}
\]

It means we work with the problem (3) that corresponds to the value function $V(l, h, t)$, which is defined as

\[
V(l, h, t) = \max_{(\pi, c)} \left[ \int_t^\infty \overline{\Phi}(t) U(c(t)) dt \mid L(t) = l, H(t) = h \right]. \tag{5}
\]

The value function $V(l, h, t)$ satisfies the Hamilton–Jacobi–Bellman (HJB) equation for the value function, in terms of $l$ and $h$

\[
V_l(l, h, t) + \frac{1}{2} \eta^2 h^2 V_{hh}(l, h, t) + (r + h)V_t(l, h, t) + (\mu - \delta) h V_h(l, h, t) + \max_{\pi} G[\pi] + \max_{c > 0} H[c] = 0, \tag{6}
\]

where $\pi \in \mathbb{R}$, $h$ is a riskless rate, $\delta$ is an expected return of risky asset, $\eta$ is the risk aversion, $\omega$ is a volatility, $G[\pi]$ is a risk premium of risky asset

\[
G[\pi] = \frac{1}{2} \sigma^2 \pi^2 + 2V_h(l, h, t) \eta \pi \sigma h + \pi(\alpha - r) V_t(l, h, t), \tag{7}
\]

Recent results

Under certain conditions we show the existence of the viscosity solutions of (4), using the internal symmetry of the leading equation we reduced the three-dimensional case (6) to a two-dimensional one or in some special cases to an ordinary differential equations. We provide the complete Lie group analysis of (4) and describe all possible reductions to lower dimensional PDEs. On the Fig. 1 one can see the results of the numerical simulation for consumption and investment strategies that we run for a Weibull and exponential case.

It is important to note that the optimal policies significantly differ from Merton solution when illiquidity becomes higher. Already when an amount of illiquid asset is more than 5% of the portfolio value the percentage of capital that is not invested in a risky stock is higher than in Merton model.
Figure 1: Consumption stream $c$ and the share of liquid capital $\pi$ stored in a risky asset depending on the ratio between the liquid and illiquid asset. As illiquid asset value becomes infinitely small the policies tend to Merton policies for a two-asset problem. We used the following parameters for assets $r = 0.01, \alpha = 0.06, \sigma = 0.5, \delta = 0.02, \rho = 0.4, \mu = 0.05, \eta = 0.3$, for Exponent distribution $\kappa = 0.2$, for Weibull parameters $\lambda = 2, k = 1$.

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