ADI time-stepping for the uncertain correlation Black–Scholes PDE

Abstract

We numerically solve the 2D uncertain correlation Black–Scholes problem using an Alternating Direction Implicit (ADI) scheme of predictor-corrector (Douglas) type, which provides highly efficient computations.

Statement of the problem

For the numerical valuation of two-asset options we consider a 2D nonlinear degenerate convection-diffusion-reaction problem with a cross derivative term. This term arises from the correlation between the two underlying stochastic processes for the asset prices and yields the nonlinearity:

\[ \frac{\partial u}{\partial t} = \frac{1}{2} \left( \sigma_1 s_1^2 \frac{\partial^2 u}{\partial s_1^2} + 2\rho(\Gamma)\sigma_1 \sigma_2 s_1 s_2 \frac{\partial^2 u}{\partial s_1 \partial s_2} + \sigma_2^2 s_2^2 \frac{\partial^2 u}{\partial s_2^2} \right) + ru \left( s_1 \frac{\partial u}{\partial s_1} + s_2 \frac{\partial u}{\partial s_2} \right) - ru \text{ for } (s_1, s_2, t) \in \Omega \times (0, T], \]

\[ \rho(\Gamma) = \begin{cases} \rho^- & \text{if } \Gamma > 0, \\ \rho^+ & \text{if } \Gamma < 0, \end{cases} \quad \Gamma = \frac{\partial^2 u}{\partial s_1 \partial s_2}, \quad \Omega = (0, S_{\max})^2. \]

We select \( u(s_1, s_2, 0) = (s - K_1)^+ - 2(s - K)^+ + (s - K_2)^+ \)
with \( s = \max(s_1, s_2) \) and \( K = (K_1 + K_2)/2 \) which is a butterfly payoff\(^1\). The boundary conditions are homogeneous on the far field boundaries and natural on the degenerate boundaries.

The numerical scheme

To alleviate the nonsmoothness of the payoff at the strikes \( K_1, K_2 \) we construct a suitable rectangular spatial grid. For each coordinate \( s_j \) a uniform mesh is taken in \([\frac{1}{2} K_1, \frac{1}{2} K_2]\) and outside this interval the mesh sizes gradually increase.

Next a second-order central finite difference discretization is applied for all derivatives in the PDE w.r.t. \( s_1 \) and \( s_2 \). This leads to a large stiff nonlinear ODE system,

\[ U'(t) = F(U(t)) = A(U(t))U(t), \quad 0 < t \leq T, \]

with given matrix function \( A(\cdot) \) and initial vector \( U(0) \equiv U_0 \).

Splitting the function \( F = F_0 + F_1 + F_2 \) with \( F_0(\xi) = A_0(\xi)\xi \), \( F_1(\xi) = A_1(\xi), \) \( F_2(\xi) = A_2(\xi) \), the Modified Craig–Sneyd (MCS) scheme \([1, 2]\) then generates approximations \( U_n \) to \( U(n\Delta t) \) for \( n \in \mathbb{N} \).

It is modern time-stepping scheme of ADI type where the (nonlinear) cross derivative part \( A_0 \) is conveniently treated in an explicit manner and for \( j = 1, 2 \) the (essentially one-dimensional) part \( A_j \) in the \( s_j \)-direction in an implicit manner.

The implicit stages stabilize the explicit stages.


Numerical results

We take \( K_1 = 35, K_2 = 45, T = 0.5, S_{\max} = 200, \sigma_1 = \sigma_2 = 0.5, r = 0.05, \rho^- = 0.4, \rho^+ = 0.6 \). Fig.1 (top) shows the numerical results for the option value at \( t = T \) and Fig.1 (bottom) for the corresponding cross derivative.

The MCS scheme and similar ADI schemes are highly efficient compared to common un-split schemes. Our present research is devoted to studying their fundamental properties, notably stability, monotonicity and convergence.

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References


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