3.3. CUTTING PLANE METHODS

Consider a pure integer linear programming problem in which all parameters are integer. This can be accomplished by multiplying the constraint by a suitable constant. Because of this assumption, also the objective function value and all the "slack" variables of the problem must have integer values.

We start by solving the LP-relaxation to get a lower bound for the minimum objective value. We assume the final simplex tableau is given, the basic variables having columns with coefficient 1 in one constraint row and 0 in other rows. The solution can be read from this form: when the nonbasic variables are 0, the basic variables have the values on right hand side (RHS) The objective function row is of the same form, with its basic variable $f$.

If the LP-solution is fractional i.e. not integer, at least one of the RHS values is fractional. We proceed by appending to the model a constraint that cuts away a part of the feasible set so that no integer solutions are lost.

Take a row $i$ from the final simplex tableau, with a fractional RHS $d$. Denote by $x_{jo}$ the basic variable of this row and $N$ the index set of nonbasic variables.

Row $i$ as an equation:

$$x_{jo} + \sum_{j \in N} w_{ij} x_j = d$$

Denote by $\lfloor d \rfloor$ the largest integer that is $\leq d$ (the whole part of $d$, if $d$ is positive). Because all variables are nonnegative,

$$\sum_{j \in N} \lfloor w_{ij} \rfloor x_j \leq \sum_{j \in N} w_{ij} x_j$$

$$\Rightarrow$$

$$x_{jo} + \sum_{j \in N} \lfloor w_{ij} \rfloor x_j \leq d$$

Left hand side is integer

$$\Rightarrow$$

$$x_{jo} + \sum_{j \in N} \lfloor w_{ij} \rfloor x_j \leq \lfloor d \rfloor$$

From the first and last formula it follows that

$$d - \lfloor d \rfloor \leq \sum_{j \in N} (w_{ij} - \lfloor w_{ij} \rfloor) x_j$$

If we denote the fractional parts by symbols

$$r = d - \lfloor d \rfloor$$

$$f_{ij} = w_{ij} - \lfloor w_{ij} \rfloor$$

we get a cut constraint or a cutting plane in the solution space:

$$\sum_{j \in N} f_{ij} x_j \geq r.$$
This equation is of basic form, with basic variable $s_i = -r$.
The resulting simplex tableau is optimal but infeasible, and we apply the dual simplex method
until all variables are nonnegative.

The cut constraints do not cut out any feasible integer points and they pass through at least one
integer point.

The next cutting plane algorithm operates with a simplex tableau.

**CUTTING PLANE ALGORITHM:**

<table>
<thead>
<tr>
<th>Step</th>
<th>Description</th>
</tr>
</thead>
</table>
| 0.   | Solve the LP-relaxation.  
      | Stop, if all variables in the solution have integer values: then it is the optimum.  
      | Stop, if the problem is infeasible or unbounded. |
| 1.   | Select a row with a fractional RHS: this is called the source row. Generate the cut constraint associated with this row:  
      | $- \sum_{j \in N} f_j x_j + s_i = -r$ |
| 2.   | Augment the simplex tableau with a column for $s_i$ and the new constraint row. |
| 3.   | Solve the LP-problem with the dual simplex method.  
      | Stop, if all variables in the solution have integer values: then it is the optimum.  
      | Stop, if the problem is infeasible or unbounded.  
      | Else, go to 1. |

The source row can be chosen arbitrarily. Also the objective row can be used because the value
of $f$ must be integer.

If a cut constraint becomes inactive again with $s_i > 0$, then variable $s_i$ and its row can be
eliminated from the tableau. This means a new cut makes the old one unnecessary. The eli-
mination restricts the growth of the simplex tableau.

The number of iterations is difficult to estimate but being an exact method like the B&B, the
computation time is not polynomially bounded.

**Example 3.2.** (cf. B&B solution in example 3.1)  

\[
\begin{align*}
\min f &= 4x_1 + 5x_2 \\
    x_1 + 4x_2 &\geq 5 \\
    3x_1 + 2x_2 &\geq 7 \\
    x_1, x_2 &\geq 0, \text{ both integer.}
\end{align*}
\]

We transform the model in a basic form using slack variables $x_3, x_4$. The LP-relaxation is then
solved with a dual simplex method as shown in the following table:
\[
\begin{align*}
&f - 4x_1 - 5x_2 = 0 \\
&-x_1 - 4x_2 + x_3 = -5 \\
&-3x_1 - 2x_2 + x_4 = -7
\end{align*}
\]

<table>
<thead>
<tr>
<th></th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
<th>( x_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f )</td>
<td>-4</td>
<td>-5</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( x_3 )</td>
<td>-1</td>
<td>-4</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>( x_4 )</td>
<td>-3</td>
<td>-2</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
<th>( x_4 )</th>
<th>( s_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f )</td>
<td>0</td>
<td>-7/3</td>
<td>0</td>
<td>-4/3</td>
<td>28/3</td>
</tr>
<tr>
<td>( x_3 )</td>
<td>0</td>
<td>-10/3</td>
<td>1</td>
<td>-1/3</td>
<td>-8/3</td>
</tr>
<tr>
<td>( x_4 )</td>
<td>1</td>
<td>2/3</td>
<td>0</td>
<td>-1/3</td>
<td>7/3</td>
</tr>
</tbody>
</table>

This is the optimal solution of the LP-relaxation.

Using the last row as the source row:

\[
x_1 + 2/10 x_3 - 4/10 x_4 = 18/10
\]

\[
x_1 + (0 + 2/10) x_3 + (-1 + 6/10) x_4 = 1 + 8/10,
\]

resulting in a cut constraint

\[
2/10 x_3 + 6/10 x_4 \geq 8/10.
\]

Append this constraint to the simplex tableau in a basic form

\[
-2/10 x_3 - 6/10 x_4 + s_1 = -8/10
\]

and continue with the dual simplex method:

<table>
<thead>
<tr>
<th></th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
<th>( x_4 )</th>
<th>( s_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f )</td>
<td>0</td>
<td>0</td>
<td>-7/10</td>
<td>-11/10</td>
<td>0</td>
</tr>
<tr>
<td>( x_3 )</td>
<td>0</td>
<td>1</td>
<td>-3/10</td>
<td>1/10</td>
<td>0</td>
</tr>
<tr>
<td>( x_4 )</td>
<td>1</td>
<td>0</td>
<td>2/10</td>
<td>-4/10</td>
<td>0</td>
</tr>
<tr>
<td>( s_1 )</td>
<td>0</td>
<td>0</td>
<td>-2/10</td>
<td>-6/10</td>
<td>1</td>
</tr>
</tbody>
</table>

Using the second row as a source row:

\[
x_2 - 2/6 x_3 + 1/6 s_1 = 4/6
\]

\[
x_2 +(-1+4/6) x_3 +(0+1/6) s_1 = 0+4/6
\]
resulting in a cut constraint

\[\frac{4}{6} x_1 + \frac{1}{6} s_1 \geq \frac{4}{6}.\]

Equation to be added to the simplex tableau:

\[-\frac{4}{6} x_3 - \frac{1}{6} s_1 + s_2 = -\frac{4}{6}\]

\[
\begin{array}{cccccc|c}
| & x_1 & x_2 & x_3 & x_4 & s_1 & s_2 | \\
\hline
f & 0 & 0 & -\frac{2}{6} & 0 & -\frac{11}{6} & 0 & 76/6 \\
x_2 & 0 & 1 & -\frac{2}{6} & 0 & \frac{1}{6} & 0 & 4/6 \\
x_1 & 1 & 0 & \frac{2}{6} & 0 & -\frac{4}{6} & 0 & 14/6 \\
x_4 & 0 & 0 & \frac{2}{6} & 1 & -\frac{10}{6} & 0 & 8/6 \\
s_2 & 0 & 0 & -\frac{4}{6} & 0 & -\frac{1}{6} & 1 & -4/6 \\
\end{array}
\]

\[
\begin{array}{cccccc|c}
| & x_1 & x_2 & x_3 & x_4 & s_1 & s_2 | \\
\hline
f & 0 & 0 & 0 & 0 & -\frac{7}{4} & -\frac{1}{2} & 13 \\
x_2 & 0 & 1 & 0 & 0 & \frac{1}{4} & -\frac{1}{2} & 1 \\
x_1 & 1 & 0 & 0 & 0 & -\frac{3}{4} & \frac{1}{2} & 2 \\
x_4 & 0 & 0 & 0 & 1 & -\frac{7}{4} & \frac{1}{2} & 1 \\
x_3 & 0 & 0 & 1 & 0 & \frac{1}{4} & -\frac{3}{2} & 1 \\
\end{array}
\]

Integer solution found: \(x_1 = 2, x_2 = 1\) ja \(f_{\text{min}} = 13\).

**COMBINATION: BRANCH AND CUT**

In the **Branch and cut method** the approaches of Branch & Bound and cutting planes are combined. Cut constraints are added to the LP-relaxation before applying the Branch&Bound. By producing tighter bounds and reducing the feasible set, the fathoming becomes more efficient, which means a smaller number of subproblems have to be generated.