Acknowledgement: A few of my students have helped in writing down my lecture notes in LaTeX. I thank Pekka Paalanen, Sapna Sharma, Vladimir X and N.N.
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1 Vector Space

1.1 What is a vector space?

Vector space is a useful mathematical structure which can represent a set of points, functions, matrices, signals and many other types of objects where the concepts addition and scalar multiples are defined. In more exact terms a vector space \( W \) is a set of objects with two operations:

\[
(x_1, x_2) \rightarrow x_1 \oplus x_2 \in W \quad \text{addition,}
\]

\[
(\alpha, x_1) \rightarrow \alpha x_1 \in W \quad \text{scalar multiplication.}
\]

Here \( x_i \in W, \alpha \in \mathbb{C} \) are arbitrary elements.

In a vector space these operations must satisfy the following eight axioms:

\[
\forall x, y, z \in W \text{ and } \forall \alpha, \beta \in \mathbb{C}:
\]

\[
x \oplus y = y \oplus x
\]

\[
(x \oplus y) \oplus z = x \oplus (y \oplus z)
\]

\[
\exists 0 \text{ s.t. } x \oplus 0 = x
\]

\[
\exists -x \text{ s.t. } x \oplus (-x) = 0
\]

\[
\alpha(\beta x) = (\alpha \beta)x
\]

\[
(1x) = x
\]

\[
\alpha(x \oplus y) = \alpha x \oplus \alpha y
\]

\[
(\alpha + \beta)x = \alpha x \oplus \beta x.
\]

For the rest of the document we will use + instead of \( \oplus \).

Examples of vector spaces:

The following are familiar examples of sets have such structure

1. \( \mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) \mid x_i \in \mathbb{R}\} \)
   \( x + y = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n) \) and \( \lambda x = (\lambda x_1, \lambda x_2, \ldots, \lambda x_n) \).

2. \( \mathcal{F}(a, b) = \{f \mid f : (a, b) \to \mathbb{R}\} \) means the function space
   \( (f + g)(x) = f(x) + g(x) \) and \( (\lambda f)(x) = \lambda f(x) \).

3. \( \mathcal{F}(A, W) = \{f : A \to W\} \), where \( W \) is a vector space

4. \( \mathcal{C}[a, b] = \{f \in \mathcal{F}[a, b] \mid f \text{ continuous}\} \)
5. $s = \{(x_n) \mid x_n \in \mathbb{R}\}$, space of signals

6. $l_1 = \{(x_n) \in s \mid \sum_{n=1}^{\infty} |x_n| < \infty\}$
   $l_2 = \{(x_n) \in s \mid \sum_{n=1}^{\infty} |x_n|^2 < \infty\}$

7. $\mathbb{R}^{n \times m} = \{A \mid A \text{ is } m \times n \text{ matrix}\}$

Later we will see how new vector spaces can be constructed by different procedures from given ones (sum, product, subspace etc).

1.2 Basic Concepts

Some basic vocabulary and concepts are listed below

- **A subspace** is a subset $M \subset W$ such that
  1. $u, v \in M \Rightarrow u + v \in M$
  2. $u \in M, \lambda \in \mathbb{R} \Rightarrow \lambda u \in M$

- **A hyperplane** is a maximal subspace
  If $W = \mathbb{R}^4$ then $M = \{(x_n) \in \mathbb{R}^4 \mid x_1 + x_2 + x_3 + x_4 = 0\}$ is a hyperplane
  If $W = C[a, b]$ then $M = \{f \mid \int_0^1 f(t)dt = 0\}$ also a hyperplane

- **A sum** of subspaces is defined as follows
  If $M_1, M_2$ subspaces, then $M_1 + M_2 = \{x + y \mid x \in M_1, y \in M_2\}$
  **Example:** In $C[0, 1]$:
  $M_1 = \{f \mid f = a \text{ constant}\}$, $M_2 = \{f \mid f(t) = ct\}$
  $M_1 + M_2 = \{f \mid f = a + ct, a, c \in \mathbb{R}\}$

- **span** of a set of vectors
  $\text{span} \{x_1, x_2, x_3, \ldots, x_n\} = \{\sum_{i=1}^{n} \lambda_i x_i \} \forall \lambda_i \in \mathbb{C}$

- **linear independence**
  $c_1 x_1 + c_2 x_2 + \cdots + c_n x_n = 0 \iff c_1 = c_2 = \cdots = c_n = 0$

- **dimension** of a vector space or subspace
  maximal number of linearly independent vectors in the space
1.3 Examples

Function (sub)spaces. The function space $F(a, b) = \{ f \mid f : (a, b) \to \mathbb{R} \}$ contains many important subspaces. For instance $C[0, 1]$ is the space of continuous functions and $C^2[0, 1]$ is the set of two times continuously differentiable functions on the interval $[0, 1]$. It is clear that

$$C^2[a, b] \subset C[a, b] \subset F[a, b].$$

The following notation $L^1[0, 1]$ means the set of integrable functions

$$\left\{ u : [0, 1] \to \mathbb{R} \mid u \text{ is integrable and } \int_{[0,1]} |u|d\mu < \infty \right\}$$

and similarly $L^2[0, 1]$ is the space of square-integrable functions. These and so called $L^p(\Omega)$ -spaces are explained in the chapter of measure theory.

Solution set of a differential equation. Define the following two sub-
sets

$$M = \{ u \in C^2[0, 1] \mid u'' + \alpha(x)u = \beta(x) \}$$

$$N = \{ u \in C^2[0, 1] \mid u'' + \alpha(x)u = 0 \}$$

$N$ is a vector subspace in $C^2$, $M$ is not. Here is the proof:

Since $u, v \in N$, we have

$$u'' + \alpha(x)u = 0$$

$$v'' + \alpha(x)v = 0$$

Let us check the axioms.

1. $u + v \in N$?

   $$(u + v)'' + \alpha(x)(u + v)$$

   $$= u'' + \alpha(x)u + v'' + \alpha(x)v$$

   $$= u'' + \alpha(x)(u) + v'' + \alpha(x)(v) = 0$$

   Clearly in $M$ $u + v$ would yield $2\beta(x)$, so $M$ is not a subspace.

2. $\lambda u \in N$? In $N$ we have

   $$(\lambda u)'' + \alpha(x)(\lambda u) = \lambda[u'' + \alpha(x)u] = \lambda 0 = 0,$$

   while in $M$ we have

   $$(\lambda u)'' + \alpha(x)(\lambda u) = \lambda[u'' + \alpha(x)u] = \lambda \beta(x) \neq \beta(x) \quad \forall \lambda$$

   Let $u_0$ be one particular solution to $u'' + \alpha(x)u = \beta(x)$. The general solution is then $u_0(x) + N$ (Fig. 1), a shifted version of the subspace $N$. See illustration.
1.4 Sum of subspaces

In vector space $W = C[-1, 1]$ define

$N = \{ f \in W \mid f(-1) = f(0) = f(1) = 0 \}$

$P_2 = \{ f \in W \mid f = \alpha_0 + \alpha_1 x + \alpha_2 x^2 \}$

$N$ and $P_2$ are vector subspaces. Study if $W = N + P_2$?

In $F = \mathbb{R}^{n \times n}$ define the subspaces of symmetric and antisymmetric matrices $M_1 = \{ A \in F \mid A = A^\top \}$ and $M_2 = \{ A \in F \mid A = -A^\top \}$.

Clearly $M_1$ and $M_2$ are subspaces. Show that $F = M_1 + M_2$.

In vector space $W = C[0, 1]$ define $C^+[0, 1] = \{ f \in W \mid f \geq 0 \}$

This subset is not a subspace, why?

Also the set $B = \{ x \in \mathbb{R}^n \mid x_1^2 + x_2^2 + \cdots + x_n^2 \leq 1 \}$ is not a subspace.

In the space of matrices the important set of binary matrices $M = \{ A \in \mathbb{R}^{n \times n} \mid A(i,j) = 0, 1 \}$ is not a subspace.

Direct sum of subspaces

If in a sum of subspaces $M = M_1 + M_2$ we have $M_1 \cap M_2 = \{ 0 \}$ we say that $M$ is a direct sum of $M_1$ and $M_2$ and we write

$M = M_1 \oplus M_2$

In this case every vector $v \in M$ has a unique representation as a sum $v = x + y$ where $x \in M_1$ and $y \in M_2$.

Example.

In $W = C[-1, 1]$ define

$N = \{ f \in W \mid f(-1) = f(0) = f(1) = 0 \}$

$P_2 = \{ f \in W \mid f = \alpha_0 + \alpha_1 x + \alpha_2 x^2 \}$

Using basic results from interpolation theory one finds that $N \cap P_2 = \{ 0 \}$ and so $W = N \oplus P_2$

In $F = \mathbb{R}^{n \times n}$ define $M_1 = \{ A \in F \mid A = A^\top \}$ and $M_2 = \{ A \in F \mid A = -A^\top \}$.

Study if the sum $F = M_1 + M_2$ is a direct sum.
1.5 Product space

If \( X, Y \) are vector spaces, we can construct a new vector space \( Z = X \times Y = \{(u, v) \mid u \in X, v \in Y\} \)

The operations of vector addition and scalar multiplication are defined in the natural way \((u_1, v_1) + (u_2, v_2) = (u_1 + u_2, v_1 + v_2)\) etc. This is called the product space of \( X \) and \( Y \). A trivial example is \( \mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R} \).

2 Normed Spaces

In a vector space one can often define a “distance” or “length” concept called a norm. This is a nonnegative function \( x \to \|x\| \geq 0 \) which satisfies three axioms given below. Such a space is called normed space and denoted as \((X, \|\cdot\|)\).

Axioms of a norm:

\[
\|x\| = 0 \iff x = 0 \\
\|x + y\| \leq \|x\| + \|y\| \\
\|\alpha x\| = |\alpha|\|x\|
\]

Example 1: The space \( \mathbb{R}^n \) with norm \( \|x\| = \sqrt{\sum x_i^2} \) and \( \mathbb{C}^n \) with norm \( \|z\| = \sqrt{\sum |z_i|^2} \)

Example 2: In \( \mathbb{C}^n \) or \( \mathbb{R}^n \) the following are norms \( \|x\|_1 = |x_1| + |x_2| + \cdots + |x_n| \) and \( \|x\|_\infty = \max_i |x_i| \) In general \( l_p \)-norm is defined as follows:

\[
\|x\| = \left( \sum_{j=1}^{n} |x_j|^p \right)^{1/p}
\]

Example 3: The following are spaces of sequences (or signals).

\( l^1 = \{(x_i)\mid \sum_{i=1}^{\infty} |x_i| < \infty \} \) with norm \( \|x\|_1 = \sum_{i=1}^{\infty} |x_i| \)

\( l^\infty = \{(x_i)\mid \sup_i |x_i| < \infty \} \) with norm \( \|x\|_\infty = \sup_i |x_i| \)

\( l^p = \{(x_i)\mid \sum_{i=1}^{\infty} |x_i|^p < \infty \} \) with norm \( \|x\|_p = (\sum_{i=1}^{\infty} |x_i|^p)^{1/p} \)
They can be used to model discrete data structures like digital signals, measurement time series etc.

In each of the cases mentioned above we should prove that the norm axioms are satisfied. Some of these are left as exercises. All can be found from text books of functional analysis [See ....].

The space $l^p$ is sometimes written as $l_p$, but they mean the same thing. The same notations apply to $l^\infty$, $L^p$ etc.

**Example 4:** We denoted by $C[a, b]$ the space of continuous functions of interval $[a, b]$.

Unless stated otherwise, usually this space is equipped with the sup-norm

$$
\|u\|_\infty = \sup_t |u(t)|.
$$

The symbol $C^k[a, b]$ means the space of $k$ times continuously differentiable functions.

### 2.1 Important inequalities

The following important inequalities are needed in the proof of the norm axiom 3 (Triangle inequality). They are also often used in proving results about convergence etc. These inequalities are true for arbitrary sets of real or complex numbers $x_i$ and $y_i$.

**Schwartz inequality**

$$
\sum_{i=1}^{n} |x_i y_i| \leq \left( \sum_{i=1}^{n} |x_i|^2 \right)^{1/2} \left( \sum_{i=1}^{n} |y_i|^2 \right)^{1/2}.
$$

This result has can be proved as follows. For every $\lambda$ the following is obviously true $\sum_{i=1}^{\infty} (x_i + \lambda y_i)^2 \geq 0$. This quantity can be written as a function of $\lambda$ as $A\lambda^2 + B\lambda + C \geq 0 \quad \forall \lambda$. As an upward parabola this function is always above the $\lambda$-axis. The determinant of this polynomial $\delta = B^2 - 4AC$ must be negative. This observation leads to the Schwartz inequality.

**Hölder inequality**

$$
\sum_{i=1}^{n} |x_i y_i| \leq \left( \sum_{i=1}^{n} |x_i|^p \right)^{1/p} \left( \sum_{i=1}^{n} |y_i|^q \right)^{1/q}, \quad \text{where} \quad \frac{1}{p} + \frac{1}{q} = 1
$$
Here we say that $p$ and $q$ are conjugate exponents. For instance if $p = 3$ then $q = 1/(1 - (1/3)) = 3/2$ The proof of Hölder’s inequality is based on an ingenious convexity argument [See...]. A consequence of this inequality if the following result (Triangle inequality for $l^p$-norm).

Minkowski inequality

$$\left( \sum_{i=1}^{n} |x_i + y_i|^p \right)^{1/p} \leq \left( \sum_{i=1}^{n} |x_i|^p \right)^{1/p} + \left( \sum_{i=1}^{n} |y_i|^p \right)^{1/p}, \quad p \geq 1$$

### 2.2 Norm in the product space

If $X, Y$ are normed spaces, we can construct a new normed space

$Z = X \times Y = \{(u, v) \mid u \in X, v \in Y\}$

A norm $\| (u, v) \|$ in this space can be for example

$$\|u\| + \|v\| \text{ or } \sqrt{\|u\|^2 + \|v\|^2} \text{ or } \max\{\|u\|, \|v\|\}$$

A trivial example is $\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$ with any $l_p$-norm. The notion of product space is a useful idea in formulation and analysis of mathematical models for many technical arrangements as the following examples show.

**Example** (Fig. 2). Consider a wire under tension, having some vibrations. State of the system can be described as $[u(x), \lambda] \in C[a,b] \times \mathbb{R}$ where $u(x;t)$ is the form of the wire at time $t$ and $\lambda$ is the stretching force.

![Figure 2: A wire under tension; example of a product space.](image)

**Example** (Fig. ??). A support beam of varying thickness is attached to a wall and loaded with mass distribution. Denote $x$ as vertical distance, $u(x)$ is the design, or varying thickness, $m(x)$ is the mass density of the loading, $d(x)$ is the deflection of the bar.

The systems reaction to the load means a function $[u(x), m(x)] \xrightarrow{F} d(x)$

Here $[u, m] \in C[0,1] \times L_1[0,1]$ and $F[u, m] \in C[0,1]$

Think about possible norms in these spaces.

### 2.3 Equivalent norms, isomorphic spaces

Let $X, Y$ be Normed Spaces and $T: X \to Y$ a mapping such that

$$\begin{align*}
T(x + y) &=Tx + Ty \\
T(\lambda x) &= \lambda Tx
\end{align*}$$


Such a mapping is called a linear mapping, linear transformation or linear operator (see chapter X).

Isomorphism is a bijective linear mapping between two vector spaces. When such mapping exists the two vector spaces can be seen to have an identical structure. Hence an isomorphic relationship may be used to derive useful conclusions.

Example. Let us consider two vector spaces

\[ X = \{ A \mid A \in \mathbb{R}^{2 \times 2} \} \]
\[ Y = \{ u \mid u \in C[0,1] : u(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3 \} = P_3[0,1] \]

We define a mapping \( T \) as a correspondence relationship between these spaces illustrated by the diagram

\[ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightarrow a + bx + cx^2 + dx^3 \]

This mapping \( T : X \rightarrow Y \) is obviously bijective and a linear operator. So \( X \) and \( Y \) are isomorphic vector spaces.

If both spaces \( X \) and \( Y \) possess norms, then the operator \( T \) is between two normed spaces. For instance, let us define norms as follows

For \( A \in X \) the norm is \( \| A \|_1 = |a| + |b| + |c| + |d| \)
and for \( u(t) \in Y \) we define \( \| u(t) \|_0 = \max \{ |a|, |b|, |c|, |d| \} \)

The we have a diagram

\[ (X, \| \|_1) \xrightarrow{T} (Y, \| \|_0) \]

We say that the mapping \( T \) is a "topological isomorphism" if the norms of \( x \) and \( Tx \) are comparable in the following sense. There are fixed constants \( m \) and \( M \) so that for each \( x \)

\[ m\|x\|_1 \leq \|Tx\|_0 \leq M\|x\|_1. \]

This condition defines a topological isomorphism. We will learn later (Chapter xx) that this condition also means continuity of the mapping \( T \) in both directions. If \( \| \|_0 \) and \( \| \|_1 \) are two norms in the same space satisfying \( m\|x\|_1 \leq \|Tx\|_0 \leq M\|x\|_1 \), they are said to be equivalent norms.
2.4 Isometry

Assume that we have a mapping between two norm spaces
\[ T : (X, \| \cdot \|_1) \rightarrow (Y, \| \cdot \|_2) \]

If for each \( x \) we have \( \| Tx \|_2 = \| x \|_1 \) then we say that

- \( T \) is a norm-preserving mapping, or
- \( T \) is an isometry.

**Example.** Let \( V = \mathbb{R}^n \), \( U \in \mathbb{R}^{n \times n} \) and \( U \) an orthogonal matrix \( (U^\top U = I) \).

Define a mapping \( U : V \rightarrow V \) as \( x \rightarrow Ux \). Then \( \| Ux \| = \| x \| \) for all \( x \), so we see that \( U \) is an isometry. In fact we know that such matrix multiplication means just rotation in the space \( \mathbb{R}^n \).

2.5 Ill posed problems, norm sensitive mappings

Many technical algorithms and mathematical operations have the generic form
\[ F(x) = y. \]

Here \( F \) may represent a system or an algorithm/program, \( x \) represents the input, initial values, initial state of the system and \( y \) represents the observed, measured or forecast output. In practice \( F \) may represent the (left side of) differential equation, partial differential equation, system of ODE:s, stochastic system, algebraic system of equations or a mixture of these.

Using this system model may mean computation of \( y \) when \( x \) is known or measured (direct problem) or solving \( x \) when \( y \) is known or measured (inverse problem).

When applying this model one may have errors, uncertainty, inaccuracy etc in either \( y \) or \( x \) or both. A system may exhibit sensitivity to initial values.

Some examples. Here \( F \) may refer to partial differential equation, like heat equation. Then \( x \) is an initial temperature distribution and \( y \) is the final temperature. Computing initial temperature backwards starting from the final is extremely sensitive to errors.

Extruding plastic/elastic polymer mass through an opening (a die) one wants to produce after solidification a profile of given form. The output form \( y \) is (due to elastic/plastic deformation) slightly different from the design of the die \( x \). One would need to find a form for the die to create a desired output profile. Here we have inverse problem \( y \rightarrow F^{-1}(x) \).
Similar phenomena are seen in light scattering, image enhancement, optical lence corrections etc.

In all the cases the question will be about how the norms $\|x\|$ and $\|Fx\|$ are related. When the mapping is an isomorphism (or an isometry), the situation is easy.

3 Convergence and continuity

3.1 Topological Terms

Metric space is a set possessing a distance function $d(x, y)$. This will be defined in later chapter (see XX). Normed space has a natural metric $\|x - y\|$. In metric spaces the following concepts are valid.

- **open ball**
  Let $X$ be a metric space with metric $d$ and $x_0 \in X$
  Open ball is a set $\{x \in X \mid d(x, x_0) < r\}$, $r > 0$.

- **closed ball**
  Closed ball is a set $\{x \in X \mid d(x, x_0) \leq r\}$, $r > 0$.

- **open set**
  A subset $M$ of metric space $X$ is open if it contains a ball about each of its points (it has no boundary points). See open ball.

- **closed set**
  A subset $K$ of metric space $X$ is closed if its complement in $X$ is open. See closed ball.

- **boundary (point)**
  Boundary point of set $A \subset X$ is a point that may or may not belong to $A$ and its every neighbourhood contain points belonging to $A$ and also points not belonging to $A$.

- **neighbourhood**
  Neighbourhood of point $x_0 \in X$ is any set in $X$ containing an open ball of radius $\varepsilon > 0$ ($\varepsilon$-neighbourhood).

- $\overset{\circ}{A}$ is interior of $A$,
  the largest open set contained in $A$.

- $\overline{A}$ is closure of $A$.
  Closure of $A$ is the smallest closed set containing $A$. 

• $\partial A$ is boundary of $A$.
  Boundary of set $A$ is the set of all boundary points of $A$.

• accumulation point
  Point $x_0 \in X$ is an accumulation point of set $M \subset X$ if every neighbour-
  hood of $x_0$ contains at least one point $y \in M$ distinct from $x_0$.
  $x_0$ may or may not be in $M$. Note that the union of the set of all
  accumulation points of $M$ and $M$ is the closure of $M$.
  Hence, you can construct an infinite sequence of points in $M$ that con-
  verges to an accumulation point, but the point does not necessarily
  belong to $M$.

Some implications of these definitions are the following. For an open set
all point are interior points or $A = \mathring{A}$. A closed set contains its boundary
points, hence $A = \overline{A}$. More exactly $\overline{A} = \mathring{A} \cup \partial A$.

3.2 Convergence of a Sequence
A sequence $(x_n)$ in metric space $X = (X,d)$ converges if $\exists x \in X$ such that
$$\lim_{n \to \infty} d(x_n, x) = 0.$$  
Here $x$ is the limit of the sequence: $x_n \longrightarrow x$.

In a normed space we can define $d(x, y) = \|x - y\|$, so it can be written
$$\lim_{n \to \infty} \|x_n - x\| = 0.$$

3.3 Continuity of a function
Let $X, Y$ normed spaces, $F : X \to Y$ function and $x_0 \in X$.
Function $F$ is continuous at $x_0$ if $x \to x_0 \Rightarrow F(x) \to F(x_0)$, or
$$\|x - x_0\| \to 0 \Rightarrow \|F(x) - F(x_0)\| \to 0.$$  
Stated in exact terms this means that $\forall \epsilon > 0$, $\exists \delta > 0$ so that
$$\|F(x) - F(x_0)\| \leq \epsilon$$  
for all $x$ satisfying $\|x - x_0\| \leq \delta$.

Exercise: If the function $F$ is linear and continuous at any point $a$, then
it is continuous everywhere.

3.4 Uniform Continuity
Uniform continuity means that in the above definition on continuity one can
select one $\delta$ which satisfies the condition at every point. More exactly
∀\(\epsilon > 0\), \(\exists \delta > 0\) so that for all \(x, y \in X\) \(\|x - y\| \leq \delta\) implies \(\|F(x) - F(y)\| \leq \epsilon\).

Figure 3 presents a curve that is not uniformly continuous; this happens because its slope becomes infinity at one point. Such function is \(f(x) = \text{sgn}(x)\sqrt{x}\) on interval \([-1, 1]\).

Figure 3: A curve that is not uniformly continuous.

Example. A function \(A : \mathbb{R}^n \rightarrow \mathbb{R}^n\), defined by a matrix \(A \in \mathbb{R}^{n \times n}\) by \(x \mapsto Ax\) is continuous and uniformly continuous

\[\|x - y\| \leq \delta \Rightarrow \|Ax - Ay\| \leq \epsilon\]

Exercise: If a function \(F\) between normed spaces is linear and continuous at any point \(a\), then it is uniformly continuous.

3.5 Compactness

Set \(A\) is said to be compact if every infinite sequence of points \((x_n)\), \(x_n \in A\) has a convergent subsequence, i.e., \(\exists a \in A\) so that a subsequence \((x_{n(i)}) \rightarrow a\).

An example of a compact set is closed and bounded set in \(\mathbb{R}^n\). To illustrate the idea assume that \(A \subset \mathbb{R}^2\) is a closed and bounded subset (Fig.4). We show that it is compact.

Figure 4: A compact set.

Choose a sequence \(x_n\) from this set \(A\). The set can be inscribed by a square \(Q_1\). Divide the square into four identical subsquares. One of then, call it \(Q_2\) must contain infinitely many points from the set \(\{x_n\}\). Divide this square into 4 subsquares. One of them, call it \(Q_3\) must again contain infinitely many points from the set \(\{x_n\}\). Continuing we generate a nested sequence of squares with size converging to zero. It is easy to pick a subsequence so that \(x_{n(i)} \in Q_i\). This sequence must be convergent as can be easily seen.

This observation is no longer true if the space has infinite dimension.

Example. A closed and bounded set in normed space

\[l_1 = \{(x_n) \mid \sum_{i=1}^\infty |x_n| < \infty\}\]
is defined as follows \( B = \{ (x_n) \mid \sum_{i=1}^{\infty} |x_n| \leq 1 \} = \{ x \in l_1 \mid \|x\|_1 \leq 1 \} \)

This set is not a compact. To see this, pick a sequence \( e_i \) from this set as follows. \( e_1 = (1, 0, 0, 0, \ldots), \) \( e_2 = (0, 1, 0, 0, \ldots), \) \( e_3 = (0, 0, 1, 0, \ldots), \ldots \)

Then \( (e_n) \) is an infinite sequence in \( B, \) but it does not contain any convergent subsequence. Why?

Compactness is an important property in optimization, for instance. In Figure 5 is shown a two-variable function. We are interested in finding the maximum of the function over two different sets (constraints) in the \( xy \)-plane, one is unbounded and one is bounded. The infinite set is not compact, the finite set is compact. The picture illustrates the following important theorem.

**Figure 5:** An infinite, non-compact set and a finite compact set.

**Theorem 3.1.** Assume that \( A \subset X \) is a compact set in a normed space \( X. \) Let \( F(u) \) be a continuous function \( F : X \to \mathbb{R}. \) Then \( F \) has a maximum (and a minimum) point in \( A. \)

To see this select a sequence \( u(i) \in A \) so that \( F(u(i)) \to \max F \) and use compactness to select a convergent subsequence. This theorem can be used to guarantee that certain optimization problems have a solution.

**Example.** Assume that the cost function of some control problem depends on the applied control/design/policy function \( f(t) \) according to an energy functional

\[
F(f) = \int_0^1 \Phi(t)|f(t)|^2 dt.
\]

Assume that this optimum is sought in a subset of functions \( B = \{ f \in C[0, 1] \mid \|f\| \leq 0.1 \}. \) This would be a constrained optimization problem in optimal control. Unfortunately the constraint set \( B \) in this case is not compact.

### 3.6 Convexity, finite dimensional spaces

A real valued function on a normed space \((V, \|\|)\) is called **convex** if for every \( x, y \in V \) and \( 0 \leq \lambda \leq 1 \) we have

\[
f[\lambda x + (1 - \lambda)y] \leq \lambda f(x) + (1 - \lambda)f(y).
\]

Assume that \( f(x) \) is a continuous convex function on the unit ball \( B = \{ x : \|x\| \leq 1 \} \) in the finite dimensional space \( \mathbb{R}^n. \) It can be shown, using
known results about the theory of real functions, that if $f(x) > 0$ for every $x \in B$ then the overall minimum is positive, that is $\min_B f(x) > 0$.

Using convexity argument one can show that in a normed space each finite dimensional subspace is closed. Assume that a subspace $H = \text{span} \{x_1, x_2, \ldots, x_n\}$ and a point $x$ is not in $H$. Define functions $d(t)$ and $f(t)$ on the unit cube of $\mathbb{R}^n$ as follows

$$d(t_1, t_2, \ldots, t_n) = \left\| \sum_{n=1}^{n} t_n x_i \right\|$$

and

$$f(t_1, t_2, \ldots, t_n) = \left\| x - \sum_{n=1}^{n} t_n x_i \right\|.$$ 

Both $d$ and $f$ are convex and continuous (exercise). Function $d$ attains its minimum $\Delta$ in the compact set $\{t : \|t_i\|_\infty \leq 1\}$. Because on linear independence we know that $\Delta > 0$. For an arbitrary point $t = (t_i)$ in $\mathbb{R}^n$ we must have

$$d(t_1, t_2, \ldots, t_n) = \| \sum_{n=1}^{n} t_n x_i \| \geq \max \|t_i\| \Delta.$$ 

Similarly for function $f(t)$ we can estimate

$$f(t) = \left\| x - \sum_{n=1}^{n} t_n x_i \right\| \geq \left\| \sum_{n=1}^{n} t_n x_i \right\| - \|x\| \geq \max \|t_i\| \Delta - \|x\|.$$ 

Using this one can show that $f(t)$ must have a positive global minimum $m$. This means that $x$ is an exterior point regarding the subspace $H$ with separating distance $m$. Since the complement is open $H$ must be closed.

One can also show that on a finite dimensional normed space all norms are equivalent. A related fact is that on a finite dimensional normed space every linear mapping $T$ must be continuous. None of these facts holds in infinite dimensional spaces.

### 3.7 Cauchy Sequence

Next we define a condition which could be called 'almost convergence'. Let $(V, \|\|)$ be a normed space and $x_n \in V$ a sequence.

$(x_n)$ is a Cauchy sequence if $\|x_n - x_m\| \rightarrow 0$ when $n$ and $m \rightarrow \infty$.

Later we will show that any Cauchy sequence actually is convergent but the limit point may be 'outside the space'.

**Example.** Each convergent sequence satisfies the Cauchy condition. Let $(x_n), x_n \rightarrow x_0$, be a convergent sequence.
Then $\|x_n - x_0\| \to 0$ and so $\|x_n - x_m\| = \|x_n - x_0 + x_0 - x_m\| \leq \|x_n - x_0\| + \|x_0 - x_m\| \to 0$ as $n, m \to \infty$.

Is the converse true? In $\mathbb{R}^n$ yes, but not in general. This is shown in the next example.

**Example.** Let $V = C[0,1]$, the space of continuous functions equipped with $\|\cdot\|_1$ -norm. Define a sequence of functions $(u_n)$ like in Figure 6: the middle part is located in the interval $\left[\frac{1}{2} - \frac{1}{n}, \frac{1}{2} + \frac{1}{n}\right]$ and the slope grows with $n$.

Figure 6: A sequence of functions.

The expression of the function $u_n(t)$ can be written

$$u_n(t) = \begin{cases} 0, & 0 \leq t \leq \frac{1}{2} - \frac{1}{n} \\ \frac{1}{2} - \frac{n}{4} + \frac{n}{2} t, & \frac{1}{2} - \frac{1}{n} \leq t \leq \frac{1}{2} + \frac{1}{n} \\ 1, & \frac{1}{2} + \frac{1}{n} \leq t \leq 1 \end{cases}$$

As illustrated in Figure 7

$$\|u_n - u_m\| = \int_0^1 |u_n(t) - u_m(t)| dt \to 0$$

and similarly

$$\|u_n - u_m\|_2 = \int_0^1 |u_n(t) - u_m(t)|^2 dt \to 0$$

when $n$ and $m$ grow to infinity. Hence, this is a Cauchy sequence in $V$. However, the limit of this sequence is not a continuous function (Fig. 8), so the sequence is not convergent in $V$.

One can see however that the sequence has a limit in the space $L^1[0,1]$ or $L^2[0,1]$.

Figure 7: $\|u_n - u_m\|$
4 Banach Space

A normed space \((V, \|\|)\) is called **complete** if every Cauchy sequence is convergent in \(V\). Complete normed space is called a **Banach space**.

Examples of Banach spaces are: \(\mathbb{C}^n\), \(\mathbb{R}^n\), \(\{C[a,b], \|\|_\infty\}\) In fact every finite dimensional normed space is complete and so a Banach space.

The sequence space \(l_p\) with the \(l_p\)-norm or \(l_\infty\) with the sup-norm.

Also the space \(L^p[a,b]\) of (Lebesgue) integrable functions with norm

\[
\|u\|_p = \left( \int_{a,b} |u(t)|^p dt \right)^{1/p}
\]

is a Banach space, see chapter (xxx). Proof of these facts are given in many basic textbooks of functional analysis. Why are we interested if any given set of mathematical objects is a Banach space? The reason is many facts and powerful theorems that are known to to be true in Banach spaces. Many useful mathematical tools, arguments, algorithms are ready to be applied once we know that we are working in a Banach space.

4.1 Completion

Many important normed spaces are not complete. However we know that every normed space is almost complete. These spaces can be 'fixed' to become Banach spaces. This is done by a procedure called completion. The basis of this idea is given below.

Let \(V\) be a normed space. There exists a complete normed space \(W\) such that \(V \subset W\) and closure of \(V\) is \(W\), \(\overline{V} = W\), we say: \(V\) is **dense** in \(W\).

A subset \(S \subset V\) is called dense if \(S = V\). This means that for any \(a \in V\) there exists \((x_n) \in S\) such that \(x_n \to a\).

**Example.** \(\{C[a,b], \|\|_1\}\) is not a Banach space. However \(C[a,b] \subset L^1[a,b]\), \(C\) is dense in \(L^1\) and \(L^1\) is a Banach space. Here \(L^1\) is the completion of \(C\).

**Example.** The space \(V = C^\infty[a,b]\) of infinitely many times differentiable functions.

For these functions so called Sobolev norms are defined as

\[
\|u\|_{n,p} = \left[ \int_a^b \sum_{i=0}^n |D^i u(t)|^p dt \right]^{1/p}
\]
\((C^\infty[a, b], \| \cdot \|_{n,p})\) is not in general a Banach space.

The above mentioned spaces and Sobolev norms are important in the theory of PDE:s. The simplest Sobolev norm is

\[
\|u\| = \int_a^b |u(t)| + |u'(t)|dt
\]

The significance of this norm can be seen by the following example. In surface inspection of a machining process or quality evaluation of newly built road we are measuring the difference between and ideal curve \(f_0(t)\) and the real result \(f(t)\). See figured below. Compare the result when (a) \(L_1\)-norm or (2) Sobolev-norm is used.

**Example.** The following Banach space might appear in CFD or other engineering application of PDE:s. \(\Omega \subset \mathbb{R}^n\), ”interior of some pressurised vessel”.

\(C^k(\Omega) = \{u \mid \text{function } u \text{ has continuous derivatives up to degree } k\}\)

\(\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)\) is a multi-index (vector of integers).

|\(\alpha| = \sum |\alpha_i|

\[
D^\alpha u = \frac{\partial^{\alpha_1} u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_n^{\alpha_n}}
\]

Define

\[
\|u\| = \max_{|\alpha|} \{|D^\alpha u|_\infty\}.
\]

Then space \((C^k(\Omega), \| \cdot \|)\) is a Banach space.

### 4.2 Contraction Theorem

We present an example of the power of Banach space theory. Let \((V, \| \cdot \|)\) be a normed space and \(F : V \to V\) a function.

\(F\) is a **contraction mapping** if \(\exists k, 0 \leq k < 1\) such that \(\|F(x) - F(y)\| \leq k\|x - y\|\) for all \(x, y \in X\).

With a contraction mapping, whatever \(x\) and \(y\) you choose, their images are closer together than the original points, by a factor less than 1.

**Theorem 4.1.** Let \(X\) be a Banach space and \(F : X \to X\) a contraction mapping.

Then there exists \(x^* \in X\) such that \(F(x^*) = x^*\) called a fixed point.

Contraction theorem has some important applications. This is illustrated by the following
Example. We study a general differential equation with initial condition

\begin{align*}
\begin{cases}
  u' &= f(u, t) \\
  u(t_0) &= u_0
\end{cases}
\end{align*}

(1)

Transform this equation into equivalent integral equation

\[ u(t) = u_0 + \int_{t_0}^{t} f(u(s), s) \, ds \]

We define a space of functions (with suitable radius a, not specified here) 
\( X = \mathcal{C}[t_0-a, t_0+a] \) with the sup-norm and an operator \( \mathcal{F} : X \to X \) by the following formula

\[ \mathcal{F}(u) = u_0 + \int_{t_0}^{t} f(u(s), s) \, ds \]

The following equivalence is obvious

Eq. 1 \( \iff u = \mathcal{F}u \).

Solving the differential equation has been transformed into a question about a fixed point of the operator \( \mathcal{F} \). Existence of the solution is guaranteed for a large class of differential equations. Whenever the kernel-function \( f(u, t) \) is such that the integral operator becomes a contraction, the solution exists by the contraction principle.

Fig. 9.

Figure 9: An example of what?

4.3 Separability

Normed space \((V, \|\cdot\|)\) is separable \(\iff\) there exists a denumerable (=countable) dense subset of \(V\). Separability is an important property which may be used in the study of convergence, approximation, numerical algorithms etc.

The set of rational numbers \(\mathbb{Q}\) is denumerable, \(\mathbb{R}\) is not. The first claim is seen by contracting an infinite table of integer pairs \((n,m)\). Every rational number \(q = n/m\) and hence has a place in this table. The cells of this table can be easily numerated \((1, 1) \to (1, 2) \to (2, 1) \to (3, 1) \to (2, 2) \to (1, 3) \to (1, 4) \to (2, 3) \cdots\).
The second claim is proved by contradiction using famous diagonal argument. Assume that the reals \([0, 1)\) can be enumerated - ordering them into a sequence \(x_n\). We represent each of them by its binary decimal representation

\[
x_1 = 0.\alpha_1^1\alpha_2^1 \ldots \alpha_m^1 \\
x_2 = 0.\alpha_1^2\alpha_2^2 \ldots \alpha_m^2 \\
x_3 = 0.\alpha_1^3\alpha_2^3 \ldots \alpha_m^3 \\
\vdots \\
x_n = 0.\alpha_1^n\alpha_2^n \ldots \alpha_m^n
\]

Then we can construct \(z = 0.(1 - \alpha_1^1)(1 - \alpha_2^2)(1 - \alpha_3^3)\ldots\) that is not found from the sequence \((x_n)\). This is a contradiction.

Examples. It is clear that the set of rationals \(\mathbb{Q}\) is dense in \(\mathbb{R}\) so it is trivially a separable space.

How about \(l_1 = \{(x_n) \mid \sum_1^\infty |x_n| < \infty\}\)? Is this a separable space?

Let \(S = \{x \mid x_i \in \mathbb{Q}\} \subset l_1\) be the set of rational sequences? It is easy to see that this set is dense in \(l_1\). Exercise! However \(S\) is not denumerable. Think binary sequences of 0:s and 1:s so \(S\) contains more elements that interval \([0, 1)\). A binary signal \(\alpha = (\alpha_0, \alpha_1, \alpha_2, \ldots)\), \(\alpha_i \in \{0, 1\}\) can be made to correspond the binary number \(a = 0.a_0a_1a_2\ldots \) \(0 \leq a < 1\) so we get a mapping \([0, 1) \leftrightarrow \) binary signals.

However the argument can be modified in a nice way

Let \(S_0 = \{(r_1, r_2, r_3, \ldots, r_k, r_{k+1} = 0, r_{k+2} = 0, \ldots), r_i \in \mathbb{Q}\}\) be the set of truncated rational sequences.

Countable union of countable sets is countable (Exercise!) and so \(S_0\) is denumerable.

\(S_0\) is also dense in \(l_1\). This is seen as follows

Figure 10

Choose arbitrary \(x = (x_i) \in l_1\), and \(\varepsilon > 0\). From the definition of \(l_1\):

\[
\sum |x_i| < \infty.
\]

We choose \(N\) big enough so that \(\sum_N^\infty |x_i| < \frac{\varepsilon}{2}\).

Next we choose \(r_1, r_2, \ldots, r_N\) so that \(\sum_1^N |r_i - x_i| < \frac{\varepsilon}{2}\)

We have constructed a vector \(r = (r_1, \ldots, r_k, 0, 0, \ldots)\) for which \(\sum_1^\infty |r_i - x_i| = \sum_1^N + \sum_N^\infty < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\)
We have constructed a vector \( r \in S_0 \) that is arbitrarily close to \( x \in l_1 \).

Figure 10: A vector of space \( l_1 \).

### 4.4 Compact Support

Let \( F \) be a function on \( X \). The support of the function is defined as

\[
\text{supp } F = \{ x \in X \mid F(x) \neq 0 \}
\]

\( F \) is said to have a compact support \( \iff \text{supp } F \) is compact. Such functions are important for instance in the study on PDE:s, FEM-methods, weak solutions etc. where so called test functions are often taken from this class.

The symbol \( C_0^0(\mathbb{R}) \) is used to denote the space of continuous functions of compact support. The space \( C_0^\infty(\mathbb{R}) \) is an important space of smooth (infinitely differentiable) compactly supported functions appearing in the theory of FEM methods, distributions, generalized derivatives etc.

### 5 Metric Spaces

There are situations where an different idea of measuring difference, distance etc is appearing, more general than norm. Let \( X \) be a set equipped with a two place function \( x, y \in X \mapsto d(x, y) \).

This function \( d \) is called a metric if \( \forall x, y, z \in X \)

1. \( d(x, y) = 0 \iff x = y \)
2. \( d(x, y) = d(y, x) \)
3. \( d(x, z) \leq d(x, y) + d(y, z) \).

**Example.** Let \( W \) be a normed space. Then \( d(x, y) = \| x - y \| \) is a metric.

\( U \subset W \), then \( (U, d) \) is a metric space, but not necessarily a vector space.

**Example.** Let \( C^+[0, 1] = \{ f \in C[0, 1] \mid f \geq 0 \} \)

Define \( \lambda(A) = "\text{the length of } A" \)

Define a distance by \( d(f, g) = \lambda \{ x \mid f(x) \neq g(x) \} \). Figure 11.

Figure 11: An example of a metric in function space.

**Example.** Let \( \Phi[a, b] = \{ f \mid [a, b] \rightarrow \mathbb{R} \mid 0 \leq f \leq 1, \text{integrable} \} \)

\[
d(f, g) = \int_a^b \min \{ |f(x) - g(x)|, \varepsilon \} dx
\]
In this example we are evaluating difference between two functions by applying a certain threshold. If the functions differ by at least $\epsilon$ then it does not matter how much the difference is. Figure 12.

Figure 12: An example of a metric in function space.

**Example.** We define $BV[0,1] = \{ f \mid f \text{ has bounded variation} \}$ and

$$\text{Var } F = |f(0)| + \sup_D \sum |f(x_{i+1}) - f(x_i)|,$$

where $D = \{x_0, x_1, \ldots, x_N\}$ is an arbitrary subdivision of the interval $[0, 1]$.

Figure 13, "length of the curve”.

A function of unbounded variation is $\sin \frac{1}{x}$ for example. Now $d(f,g) = \text{Var}(f-g)$ is a metric in $BV[0,1]$.

Figure 13: Piecewise linear approximation of curve length.

This space $BV[0,1]$ has some theoretical interest because it can be shown to be the dual space of $L^\infty[0,1]$. See a later chapter about duality.

### 5.1 Translation Invariance

Let $V$ be a vector space. If there exists a metric $d$, does

$$d(x,y) = d(x + a, y + a)$$

hold? Generally no, but if the metric is defined through a norm, we can calculate

$$d(x,y) = \|x - y\| = \|x + a - y - a\| = d(x + a, y + a)$$

We see that a norm always produces a translation invariant metric.

### 5.2 Application areas

One example of using metric in technical applications is image recognition. Think a gas station or railway station where you can pay by a bank note. The machine must recognize if the bank note is real. The recognition is based on comparison of the measured signal, the digital image of the received bank
note, with a stored official image of a real bank note. These two images are not exactly identical (why?) and so the machine needs to compute the distance between the two images. So here we need a metric to compare two images. Other application areas would be

- evaluating the convergence of a numerical algorithm by measuring a distance between two functions
- in digital codes to define a distance between code words (especially error correction codes)
- comparing a measured heart curve EKG with an ideal curve of a healthy person to diagnose the condition of the heart
- measuring similarity of text samples in comparing authors’ styles
- measuring similarity/dissimilarity of fuzzy sets.

6 Hilbert Space

An important class of normed spaces are those where the norm is generated by an inner product.

Let $H$ be a vector space. Mapping $H \times H \rightarrow \mathbb{R}$ or $\mathbb{C}$ denoted by symbol $(x, y) \mapsto \langle x, y \rangle$ is an inner product if it satisfies the following axioms:

1. $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$
2. $\langle x, y \rangle = \overline{\langle y, x \rangle}$ (complex conjugate)
3. $\langle x, x \rangle \geq 0, \quad \langle x, x \rangle = 0 \iff x = 0$

Inner product generates a norm $\|x\| = \sqrt{\langle x, x \rangle}$ is a norm. This satisfies the Schwartz inequality.

$$|\langle x, y \rangle| \leq \|x\| \|y\|$$

The proof is an easy adaptation of the proof presented for vectors in $\mathbb{R}^n$.

We call $(H, \langle , \rangle)$ an inner product space. If $H$ is complete, that is if every Cauchy sequence is convergent in $H$, we say that $H$ is a Hilbert space. Following inclusions hold Hilbert $\subset$ Banach $\subset$ normed $\subset$ vector spaces.
Examples. The spaces $\mathbb{R}^n$, $\mathbb{C}^n$, equipped with
\[ \langle x, y \rangle \equiv x_1y_1 + x_2y_2 + \cdots + x_ny_n \]
are Hilbert spaces.

$C[a, b]$ with $\langle u, v \rangle \equiv \int_a^b u(t)v(t)dt$ is an inner product space, but not a Hilbert space.

The space $l_2$ with $\langle u, v \rangle \equiv \sum_1^\infty x_ny_n$ is the only $l_p$-type Hilbert space. In fact practically all Hilbert spaces are isomorphic to $l_2$.

The function space $L^2(a, b) = \{ f | \int_a^b |f(t)|^2 dt < \infty \}$ with inner product $\langle u, v \rangle \equiv \int_a^b u(t)v(t)dt$ is a Hilbert space.

Several other inner products can be defined. For instance an inner product with weight function $\langle u, v \rangle_\phi \equiv \int_a^b \phi(t)u(t)v(t)dt, \ \phi(t) > 0$ or

The following inner product is an example from Sobolev spaces. $\langle u, v \rangle_S \equiv \int_a^b u(t)v(t)dt + \int_a^b DuDv(t)dt$ valid in $C[a, b] \cap L^2(a, b)$

6.1 Properties of Inner Product

The following note is often useful
\[ \langle x, y \rangle = 0 \ \forall x \Rightarrow y = 0 \]

Parallelogram law. The Following is true in all inner products \( \|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2 \), Figure 14.

Figure 14: Sum and difference of two vectors.

Continuity of norm and inner product. Let $x_i$ be a sequence in vector space $V$. If $x_i \to x$, then $\|x_i\| \to \|x\|$. If in a Hilbert space $x_i \to x$ and $y_i \to y$ then one can easily prove $\langle x_i, y_i \rangle \to \langle x, y \rangle$. Think Schwarz inequality.

Infinite sums. If \( \{x_i\} \) is a sequence in normed space $V$ and the finite sums $\sum_1^k x_i$ form a convergent sequence, that is $\sum_1^k x_i \to x$ or $\| \sum_1^k x_i - x \| \to 0$ we write $\sum_1^\infty x_i = x$ and say that the series is convergent.

If we have a convergent series $\sum_1^\infty x_i = x$ in a Hilbert space $H$ and $z \in H$, then using the continuity of the inner product we can write
\[ \langle \sum_1^k x_i, z \rangle = \sum_1^k \langle x_i, z \rangle \to \langle x, z \rangle \]
and so
\[ \langle x, z \rangle = \sum_{1}^{\infty} \langle x_i, z \rangle \]

### 6.2 Orthogonality

Two vectors are **orthogonal** if their inner product is zero.

\[ x \perp y \iff \langle x, y \rangle = 0 \]

The Pythagorean theorem holds for inner product spaces.

\[ x \perp y \Rightarrow \| x + y \|^2 = \| x \|^2 + \| y \|^2 \]

Define **orthogonal complement** of \( S \) by \( S^\perp = \{ x \in H \mid x \perp y, \forall y \in S \} \). \( S^\perp \) is always a closed subspace (Exercise!). Also it is not difficult to see that \( S^{\perp \perp} = \text{span} S \). The notation refers to the closure of \( \text{span} S \).

### 6.3 Projection Principle

Let \( S \) be a closed subspace in a Hilbert space \( H \). Let \( x \in H \), but \( x \notin S \). One can prove that there exists a unique point \( y \in S \) such that

\[ \| x - y \| = \min \{ \| x - z \| \mid z \in S \} \]

Figure 15.

Figure 15: Projecting \( x \) onto \( S \).

This point of minimal distance is found by orthogonal projection: find \( y \) such that \( (x - y) \perp S \) or \( \langle x - y, z \rangle = 0 \forall z \in S \).

**Example.** Approximate the function \( f(t) = \sin t^2 \) on an interval \([a, b]\) with a polynomial of degree \( n \). Here \( f(t) \in L^2(a, b) \) and we define a subspace

\[ S = \text{span} \{ 1, t, t^2, \ldots, t^n \} \]

Application of the projection principle means the following task. Solve coefficients \( \alpha_i \) from

\[ \int_{a}^{b} [f(t) - (\alpha_0 + \alpha_1 t + \alpha_2 t^2 + \cdots + \alpha_n t^n)] t^k dt = 0 \]

for \( k = 0, 1, 2, \ldots, n \). This system of equations gives the optimal approximation.
6.4 Projection Operator

The operator $P_S : H \to S$ that maps a vector $x \in H$ to the orthogonal projection vector $y \in S$ is called projection operator. (Figure 15).

It is clear that this operator is linear. Moreover $P_S \cdot P_S = P_S$ so that projection operator satisfies equation $P^2 = P$.

**Example.** Consider the Moving average operator $M$ in signal filtering. Obviously $M^2 \neq M$ and so it is not a projection.

6.5 Orthogonal Sequence

Let $(u_i)$ be a sequence, $u_i \in H$. Orthogonal sequence is defined as

$$u_i \perp u_j \text{ or } \langle u_i, u_j \rangle = 0 \quad \forall i \neq j$$

Take $\{u_1, u_2, \ldots, u_k\}$ orthogonal. Then arbitrary vector $x$ in space spanned by $u_i$ can be represented as $x = \sum_1^k \alpha_i u_i$.

Take an inner product of $x$ with each of $u_j$.

$$\langle x, u_j \rangle = \sum_1^k \alpha_i \langle u_i, u_j \rangle = \alpha_j \langle u_j, u_j \rangle$$

$$\Rightarrow \quad \alpha_j = \frac{\langle x, u_j \rangle}{\langle u_j, u_j \rangle}$$

Let $(u_j)$ be an infinite orthogonal sequence. Often one can represent a vector as a sum of an infinite series as follows

$$x = \sum_1^\infty \alpha_i u_i$$

This means that the finite partial sums

$$x^{(k)} = \sum_1^k \alpha_i u_i$$

converge to $x$ in the norm of the space

$$\|x - x^{(k)}\| \xrightarrow{n \to \infty} 0.$$
\[ \langle x, u_j \rangle = \left\langle \sum_1^\infty \alpha_i u_i, u_j \right\rangle = \sum_1^\infty \alpha_i \langle u_i, u_j \rangle \]

also in the case of infinite series. For finited sums it is clearly true. The prove this for infinite sums one needs to use the fact that norm \( x \to \|x\| \) and inner product \( x, y \to \langle x, y \rangle \) are continuous functions meaning that if \( x_n \to x_0 \) and \( y_n \to y_0 \) then
\[
\|x_n - x_0\| \to 0
\]
and
\[
\langle x_n, y_n \rangle \to \langle x_0, y_0 \rangle.
\]
Hence we know that the formula given earlier about the coefficients is also valid for infinite orthogonal series
\[
\alpha_j = \langle x_j, u_j \rangle / \langle u_j, u_j \rangle.
\]

6.6 Orthonormal Sequence

Sequence \( \{u_i\} \) is orthonormal if
\[
u_i \perp u_j \text{ when } i \neq j \text{ and } \|u_i\| = 1 \text{ for all } i.
\]
An equivalent condition is
\[
\langle u_i, u_j \rangle = 0, \quad i \neq j
\]
\[
\langle u_i, u_j \rangle = 1, \quad i = j
\]
The infinite series representation \( x = \sum_1^\infty \alpha_i u_i \) is easy to compute
\[
\langle x, u_j \rangle = \ldots = \alpha_j \langle u_j, u_j \rangle = \alpha_j
\]
We get the Fourier representation
\[
x = \sum_1^\infty \langle x, u_i \rangle \, u_i
\]
6.7 Maximal Orthonormal Sequence

Let $H$ be a Hilbert space and $\{u_i\}$ an orthonormal sequence. If $\text{span}\{u_i\} = H$

then we say $\{u_i\}$ is maximal orthonormal sequence (or total orthonormal sequence).

Let $x \in H$ be an arbitrary vector. Set $y = \sum_{1}^{\infty} \langle x, u_i \rangle u_i$. We can compute as follows

$$\langle x - y, u_j \rangle = \langle x, u_j \rangle - \langle y, u_j \rangle$$

$$= \langle x, u_j \rangle - \left( \sum_{1}^{\infty} \langle x, u_i \rangle u_i, u_j \right)$$

$$= \langle x, u_j \rangle - \sum_{1}^{\infty} \langle x, u_i \rangle \langle u_i, u_j \rangle$$

$$= \langle x, u_j \rangle - \langle x, u_j \rangle = 0.$$ 

If follows that for all linear combinations we have

$$\left\langle x - y, \sum_{1}^{k} c_i u_i \right\rangle = 0.$$ 

These linear combinations are dense in $H$ so we have $\langle x - y, z \rangle = 0 \ \forall z$

We remember the Hilbert space axiom: $\langle v, z \rangle = 0 \ \forall z \in H \Rightarrow v = 0$ .

Hence $x = y$ and we see that

$x = \sum_{1}^{\infty} \langle x, u_i \rangle u_i$ exists for every $x \in H$.

6.8 Orthogonal Polynomials

Define inner product in a Hilbert function space as

$$\langle u, v \rangle_M = \int_{a}^{b} M(t)u(t)v(t)dt.$$ 

The weight function $M(t) > 0$, except at the end points it can be also zero.

For many choises of weight functions $\{L^2(a,b), \langle , \rangle_M\}$ is a Hilbert space. Starting from the usual polynomials $t^n$ and applying the well known Gramm–Schmidt– process one can generate an orthonorma sequence with respect to the inner product $\langle , \rangle_M$. 

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\{1, t, t^2, \ldots, t^n, \ldots\} \xrightarrow{\text{Gram-Schmidt}} \{\Phi_n\}

For \(M(t) = 1\) we get the Legendre polynomials.
For \(M(t) = 1/\sqrt{1-t^2}\), \(t \in [-1, 1]\) we get the Chebyshev polynomials.
For \(M(t) = (1-x)^a(1+x)^b\) one gets Jacobi Polynomials.
The weight function \(M(t) = e^{-x}\) generates Laguerre polynomials. Hermite polynomials are orthogonal with respect to the Gaussian weight function
\[M(t) = e^{-x^2/2}\]  

Orthogonal polynomials are used in deriving solutions for PDEs for instance. In so called spectral methods one uses orthogonal expansions of basis functions.

6.9 Orthogonal Subspaces

Let \(H\) be a Hilbert space and \(S \subset H\) a subspace.

The spaces \(S\) and \(S^\perp\) are orthogonal subspaces and \(H = S \oplus S^\perp\). This means that for any \(x \in H\) one can find \(y \in S\) and \(z \in S^\perp\) so that \(x = y + z\).

Consider a mutual orthogonal set of subspaces \(M_1, M_2, \ldots, M_n\), where \(M_i \perp M_j, i \neq j\). Let \(P = P_i\) be the projection operator onto the subspace \(M_i\).

In case the subspaces span the whole space, we can write
\[H = M_1 \oplus M_2 \oplus \cdots \oplus M_n.\]
This is called an orthogonal decomposition of the space \(H\).

For each vector \(x\) we can write
\[x = x_1 + x_2 + \cdots + x_n, \text{ and for each } i, j, x_i \perp x_j\]
\[= P_1 x + P_2 x + \cdots + P_n x\]

This means that the identity operator has been split into a sum of projections
\[I = P_1 + P_2 + \cdots + P_n.\]

**Example.** Let \(H = L^2(-\pi, \pi)\) and \(M_n = \text{span} \{\cos nt\}\). It is known in calculus that
\[\langle \cos nt, \cos mt \rangle = \int_{-\pi}^{\pi} \cos(nt)\cos(mt)dt = 0, \text{ for all } n \neq m.\]

Hence we have a decomposition \(H = M_1 \oplus M_2 \oplus \cdots \oplus M_n \oplus H_n^\perp\).
7 Measure theory, measure spaces

If $A \in \mathbb{R}^2$ is a set, we can intuitively define a function $\mu(A) = \text{"area of } A\text{"}$. The definition is clear for a rectangle, or if the set can be approximated by combination of rectangles. However it is known that it is not possible to define this concept for arbitrary sets $A$. Area can be defined only for a collection of measurable sets: The collection of measurable sets has a structure of sigma algebra.

Let $\Omega$ be a set (a universe) and $P$ a collection of subsets $A \subset \Omega$. This collection is **Sigma algebra** if

1. $\phi \in P$
2. $A \in P \Rightarrow A^c \in P$
3. $A_i \in P \Rightarrow \bigcup_{i} A_i \in P$

A measure is a function defined on a sigma algebra. More exactly, let $\mu : \Omega \rightarrow \mathbb{R}^+$ be a function such that

1. $\mu(\phi) = 0$
2. $\mu\left(\bigcup E_i\right) = \sum_{1}^{\infty} \mu(E_i)$ if $E_i \cap E_j = 0$

We say that $\mu$ is a measure and $(\Omega, P, \mu)$ is a **measure space**.

**Example**: Here $(p_i)$ is a sequence of points in $\mathbb{R}$ and $\sharp(A) = \sharp\{i \mid p_i \in A\}$. Here $\sharp$ means the counting function and is also called “Counting measure”. If $B$ is any sigma algebra of sets in $\mathbb{R}$, then $(\mathbb{R}, B, \sharp)$ is a measure space.

**Example**: Suppose we have a random experiment, where the range of all possible outcomes is $\Omega$, a collection of possible events is a sigma algebra $P$. We define the **probability measure** by $p(A) = P\{\omega \in A\}$. Then $(\Omega, P, p)$ is a measure space, also called **probability space**.

**Example**: Let us have an amount of some material (imagine smoke in space or ink on paper, dye dissolved in liquid, electrical charge in material) distributed/dissolved in a space $\Omega$. We define a function $\mu(A) = \text{amount of substance in } A$. This defines a measure.

The material can be distributed in a continuous cloud, or in pointwise grains (like color/dye). The mass distribution can be a mixture of both
types. Electrical charge can be distributed in a mixture consisting of space
cloud, surfaces, sharp edges, or points. Singular points may have non-zero
measure $\mu(\{x\}) > 0$

Example:

Imagine the following experiment. We have a disc where one tiny section
of the perimeter has been cut so that there is a short line segment. The disc
is located in a space between two walls where it can roll easily. The disc
is perturbed from start position and rolling back and froth until it stops.
Denote by $\alpha$ the angle of rotation from start position. In this random example
the stopping angle $\alpha$ has an interesting distribution. What kind?

7.1 Lebesgue Integral

Let $(\Omega, P, \mu)$ be a measure space. A simple function $s : \Omega \rightarrow \mathbb{R}$ is defined
as $s = \sum_{i=1}^{n} c_i \chi_{E_i}$, where

$$\chi_{E}(t) = \begin{cases} 
1 & \text{if } t \in E \\
0 & \text{otherwise}
\end{cases}$$

means the characteristic function of a set $E$.

Integral of a simple function over $\Omega$ is defined as

$$\int_{\Omega} s d\mu = \sum_{i=1}^{n} c_i \mu(E_i)$$

Integral over a subset $A$ is $\int_{A} s d\mu = \sum_{i=1}^{n} c_i \mu(E_i \cap A)$

The integral over arbitrary integrable (measurable) function $u : \Omega \rightarrow \mathbb{R}$
is defined via a limit process.

**Definition.** Let $\{s_n\}$ be an increasing sequence of simple functions such
that $s_n \rightarrow \mu$ (pointwise). The limit $\int_{A} u d\mu = \lim_{n} \int_{A} s_n d\mu$ is called a lower
integral.

A function is integrable if a similar approximation by a decreasing se-
quence from above produces the same limit. Such function is called Lebesgue
integrable. Integrable functions may be non-smooth, with infinitely many
discontinuities etc. However they are very close to nice functions.
If \( u \) is integrable and \( \epsilon > 0 \) then \( \exists v \in C^\infty \) such that

\[
\int_{-\infty}^{\infty} |u - v|d\mu < \epsilon
\]

The space of Lebesque integrable functions is defined

\[
L^p(\Omega) = \{ \mu : \Omega \to \mathbb{R} | \text{integrable } \int_{\Omega} |u|^p d\mu < \infty \}
\]

The following expression looks like a norm

\[
\|u\|_p = \left[ \int_{\Omega} |f|^p d\mu \right]^{\frac{1}{p}}
\]

However the third norm axiom is not true because

\[
\|u\| = 0 \iff \mu\{x \mid u(x) \neq 0\} = 0.
\]

In this case we say that \( u = 0 \) almost everywhere.

We identify all functions that are equal almost everywhere. This is done by introducing equivalence classes \([u] = \{f \mid f = u \text{ almost everywhere}\}\).

Finally we define a space \( L^p \) as the space of equivalence classes

\[
L^p(\Omega) = \{ [u] : \int |u|^p d\mu < \infty \}
\]

This space is a banach space.

### 7.2 Riemann-Stieltjes Integral

\( \alpha : [a, b] \to \mathbb{R} \), non-decreasing function and continuous on the right.

\( f : [a, b] \to \mathbb{R} \) a measurable function.

We divide the interval \([a, b]\) into subintervals by intermediate points. This is called a partition of the interval

\( \pi = \{a = x_0 < x_1 < x_2 < \ldots < x_n = b\} \). We study the following sums taken over all partitions

\[
\sum_{i=1}^{n} f(\xi_i)[\alpha(x_{i+1}) - \alpha x_i] \to \int_{a}^{b} f(x) d\alpha(x)
\]

where \( \xi_i \in [x_i, x_{i+1}] \). If this sum has a limit when the partition is ultimately refined \{so that the \( \|\pi\| = max\|x_{i+1} - x_i\| \to 0 \)\} this limit is called the Riemann-Stieltjes integral of the function \( f(x) \) with respect to the integrator function \( \alpha(x) \).
8 Linear transformations

Let $X, Y$ be Normed Spaces and $T: X \rightarrow Y$ a mapping such that

\[
\begin{align*}
T(x + y) &= Tx + Ty \\
T(\lambda x) &= \lambda Tx
\end{align*}
\]

Such mapping is called linear transformation or linear operator.

Let $X = C[0, \infty)$, $Y = C^2[0, \infty)$. Then taking a derivative is a linear operator $D: Y \rightarrow X$ mapping $u \rightarrow u'$

Similarly $D^2: Y \rightarrow X$ mapping $u \rightarrow u''$ is a linear operator.

The following is a familiar differential operator - being the left hand side of a well-known circuit equation. $S = LD^2 + RD + \frac{1}{c} I$.

Solving the circuit equation $Lu'' + Ru' + \frac{1}{c} u = v$ is equivalent to solving operator equation $s\mu = v$.

From the theory of differential equations we know that the solution is

\[u(t) = \int_0^t k(t - s)v(s)ds\]

where

\[k = k(t) = \frac{(e^{\lambda_1 t} - e^{\lambda_2 t})}{L(\lambda_1 - \lambda_2)}\]

This integral formula $u = Tv$ defines another linear operator $T: X \rightarrow Y$.

It is clear that $S \cdot T = I$

showing that $S$ and $T$ are inverse to each other.

Examples:

The formula $L(f(t)) = \int_0^\infty e^{-st}f(t)dt$ defines a linear operator $f(t) \rightarrow \hat{f}(s)$.

This is called Laplace Transform

The formula $\Phi(f(t)) = \int_{-\infty}^\infty e^{-i\omega t}f(t)dt$ defines Fourier Transform

In signal analysis we have convolution $(f \ast g)(t) = \int_{-\infty}^\infty f(s)g(t - s)ds$

which is also a linear operation.
Linear partial differential equation PDE contains a linear operator on functions $u = u(x, t)$

In the following PDE

$$\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}[k(x)\frac{\partial u}{\partial x}] = q(x)$$

we see the action of following linear operators

$$u \rightarrow D_t u$$
$$u \rightarrow k(x) u$$
$$u \rightarrow D_x u \quad \text{and}$$
$$u \rightarrow [D_t - D_x[k(x)D_x]] \cdot u$$

**Example:** The following is a generic formula of an integral equation

In space $X = L_2(a, b)$ define an operator $T : X \rightarrow X$ by

$$(Tu)(s) = \int_a^b k(s, t) u(t) dt$$

Here $k = k(s, t) \in L^2[(a, b)x(a, b)]$ is called the kernel of the operator. Note that the transforms mentioned above are actually special cases of the generic formula.

### 8.1 Bounded Linear Transformation

A linear operator between normed space is called **bounded** if

$$\|Tx\| \leq K \cdot \|x\|$$

for some constant $K$.

It is easily seen that for a linear mapping $T$ is bounded $\iff$ Tis continuous

If $x_n \rightarrow x_0$ then $\|Tx_n - Ty_0\| = \|T(x_n - x_0)\| \leq K \cdot \|x_n - x_0\|$. The reverse implication is left as an exercise.

**Example:** Some common operators are not bounded.

Consider $D : C^1[0, \infty) \rightarrow C[0, \infty)$, with $\|\|_\infty$-norm. By studying functions $u_n(t) = 1/sqrt n \cdot sin(nt)$ one can see that $D$ is not a bounded operator.

We call an operator $T$ **invertible** if $T^{-1}$ exist and is bounded.
8.2 Norm of an Operator

The norm of an operator \( T : X \rightarrow Y \) is defined

\[
\|T\| = \sup \{\|Tx\|/\|x\| : x \neq 0\}
\]

or equivalently

\[
\|T\| = \sup \{\|Tx\|/\|x\| \leq 1\}.
\]

Then we have always \( \|Tx\| \leq \|T\|\|x\| \).

The set of operators \( L(X,Y) = \{T \mid T : X \rightarrow Y, \text{linear}\} \) equipped with this norm then becomes itself a normed space \( \{L(X,Y),\|\|\}\} \).

Example A matrix \( A \in \mathbb{R}^{m \times n} \) defines a linear operator \( A : \mathbb{R}^n \rightarrow \mathbb{R}^m \) via matrix multiplication \( x \rightarrow Ax \).

Specifying norms on both spaces we get an operator between normed spaces, such as \( A : [\mathbb{R}^n, \|\|_\infty] \rightarrow [\mathbb{R}^m, \|\|_1] \).

In this case the operator norm \( \|A\|_{\infty,1} = \sup \{\|Ax\|_1 : \|x\|_\infty \leq 1\} \) is called a Matrix Norm.

Example: When we have \( A : [\mathbb{R}^n, \|\|_2] \rightarrow [\mathbb{R}^m, \|\|_2] \) the matrix norm is known to be

\[
\|A\| = \|A\|_{2,2} = \lambda_{\text{max}}^{1/2},
\]

where \( \lambda_{\text{max}} \) = largest eigenvalue of the matrix \( A^T A \). This is used for instance in the construction of single value decomposition SVD

8.3 Composite operator

Assume that \( Y, Y \) and \( Z \) are normed spaces and we have operators \( T \) and \( S \) between them.

\[
X \xrightarrow{T} Y \xrightarrow{S} Z
\]

The the composite mapping \( ST : X \rightarrow Z \) is created and \( ST(x) = S(T(x)) \). In this situation it is clear that the following norm inequality holds \( \|ST\| \leq \|S\|\|T\| \).
8.4 Exponential Operator Function

When operator $T$ acts inside the same space, that is $T : X \to X$, then we can define the $n$:th power of the operator $T.T.T...T = T^n$. It is clear that then

$$\|T^n\| \leq \|T\|^n$$

If $A : X \to X$ is a bounded operator on $X$, we can define an exponential function $\exp(A)$ or $e^A$ of the operator $A$ as the limit of power series

$$I + A + \frac{1}{2!}A^2 + \ldots + \frac{1}{k!}A^k = T_k \to e^A.$$ 

To derive this result on convergence, the following observations are used. The space of operators $L(X) = L(X, X)$ is a Banach space if $X$ is a Banach space. Moreover for the partial sums $T_k = I + A + \ldots + \frac{1}{k!}A^k$ we can show that $\|T_k - T_m\| \to 0$ as $k,m \to \infty$. Note that we can calculate

$$\|T_k - T_m\| = \| \sum_{m}^{k} A^k / k!\| \leq \sum_{m}^{k} \|A^k / k!\| \to 0$$ since this is a section of the power series of the real exponential function $e^t$.

8.5 Filters in Signal Spaces

In digital electronics, processing video/audio signals one is often working with linear operations. In the space $X = s = \{(x_n) \mid n = -\infty, \ldots, \infty\}$ we can define linear map $T : X \to X$ as follows $T(X_n) \to T(x_{n-1})$. This is called the shift operator. It is a linear mapping in the signal space. What would mean the n:th power $T^n$? What would the following operator do $\alpha T + \beta T^2$?

Next we define another mapping \{\{X_n \to Z_n\}\} in the signal space. The following operator is called 'Moving Average filter'.

$$Z_n = \frac{1}{k+1} \sum_{0}^{k} X_{n-j}.$$ It is also a linear map.

In general the formula $Z_n = \sum_{0}^{k} a_k X_{n-j}$ represents an arbitrary filter in the signal space. The coefficients ($a_k$) will define the behavior of the filter. By a choice of suitable coefficient vector the filter may modify the signal by removing (smoothing out) certain features (high-pass, low-pass filters etc).
8.6 Linear Functional

Let $X$ be a Normed Space. A linear mapping $f : X \to \mathbb{R}$ (or in $\mathbb{C}$) linear is called linear functional.

The space $X' = \{ f \mid f$ is linear functional $\}$ is called the algebraic Dual Space of $X$.

The subspace $X^* = \{ f \mid f$ is continuous linear functional $\}$ is called the dual Space or sometimes the topological Dual Space of $X$.

The norm in this space is

$$\|f\| = \sup_{\|x\| \leq 1} |f(x)| < \infty$$

**Example:** $X = l^1, x = (x_n) \in l^1$. Examples of linear functionals in this space are

$f(x) = \sum_1^\infty x_i$

$f(x) = x_1 - x_{10}$

$f(x) = \sum_{10} x_i$.

In the function space $X = C[a, b]$ the following are linear functionals

$f(u) = \int_a^b u(t) dt$

$g(u) = \frac{1}{2} [u(a) + u(b)]$

$h(u) = \int_a^b \phi(t) u(t) dt$.

**Example:** Let $H$ be so called Haar function [see]. The following functions $u_{i,k}(x) = 2^{-i}H(2^i(x - k))$ are a basis for an orthonormal wavelet expansion

$$f(x) = \sum_{i,k} \alpha_{i,k} h_{i,k}(x)$$

The formula to compute the coefficient $\alpha_{i,k} h_{i,k}(x)$ is, as we know from Fourier theory

$$f(x) \to \int_{-\infty}^{\infty} f(x) \cdot h_{i,k}(x) dt.$$  

This formula can be seen also as a linear functional on $X = L^2(\mathbb{R})$.

**Example:** Let $X = l^n_p = (\mathbb{R}, \|\|_p)$ and $x = (x_1, x_2, \ldots, x_n) \in X$. We define a linear function $f : \mathbb{R}^n \to \mathbb{R}$ as follows

$$f(x) = \alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n = \langle a, x \rangle,$$
where \( a = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n \). It is clearly a linear functional.

A consequence of Hölder inequality is that if \( \|x\|_p \leq 1 \) then \( |f(x)| = |\langle x, a \rangle| \leq \|a\|_q \), where \( 1/p + 1/q = 1 \). One can prove that in fact

\[
\|f\| = \sup_{\|x\|_p \leq 1} |\langle x, a \rangle| = \sup_{\|x\|_p \leq 1} \left| \sum_{i=1}^{n} \alpha_i x_i \right| = \|a\|_q,
\]

with \( \left( \frac{1}{p} + \frac{1}{q} = 1 \right) \).

This means that we have identified elements of the dual, and

\[(l^n_p)^* = l^n_q.\]

One can also prove that the same holds for infinite dimensional case, that is

\[(l^n_p)^* = l_q.\]

**Example:** The case of function space \( X = L^p \) is similar and very important. One can show that in the space \( X = L^p(a, b) \) linear functionals take the form

\[F(u) = \int_a^b u(x) \cdot v(t) dt = \langle u(v), v(t) \rangle,\]

where \( v(t) \in L_q[a, b] \) and so we can say that

\([L^p(a, b)]^* = L^q(a, b).\]

To identify the dual space of a given normed space is not trivial. In the space of continuous functions \( C[a, b] \) the following formula defines a linear functional

\[f(u) = \int_a^b u(t) dF(t),\]

where \( F(t) \) is a function of bounded variation and the integral is of Riemann-Stieltjes type. Let us define \( NBV[a, b] \) to be the space of BV-functions \( F(t) \) normalized as \( F(a) = 0 \). One can show that all continuous functionals on \( C[a, b] \) are of this type and so

\[C^*[a, b] = NBV[a, b].\]
8.7 Duality, weak solutions

If $X$ and $Y$ are normed spaces and $A : X \to Y$ is a linear operator, then we can define a dual or adjoint operator $A^* : Y^* \to X^*$ between the dual spaces by formula

$$A^*g(x) = g(Ax) \text{ for all } g \in Y^*.$$  

The notation $\langle x, f \rangle$ is often used instead of $f(x)$ when speaking of linear functionals. Hence the definition of the dual operator can also be written

$$\langle x, A^*g \rangle = \langle Ax, g \rangle.$$  

A sequence $x_n$ in a normed space $X$ is said to be weakly convergent to a limit $x_0$, if

$$\langle x_n - x_0, a \rangle \to 0 \text{ for all } a \in X^*.$$  

Convergence in norm implies weak convergence but not vice versa.

As a consequence of a well-known theorem (Hahn-Banach) we know that for any vector $x \in X$ we have

$$\|x\| = \sup \{ |\langle x, a \rangle| : a \in X^* \}.$$  

A consequence of this is a that if for a vector $x$ we have $\langle x, a \rangle = 0$ for all linear functionals $a \in X^*$ then necessarily $x = 0$. If fact we know that if we have a dense subset $S \subset X^*$ then

$$\langle x, a \rangle = 0 \text{ for all } a \in S \Rightarrow x = 0.$$  

This fact is used in the derivation of so called weak solutions. Assume that we are studying an equation

$$Lx = b,$$

where $L : X \to Y$ is a linear operator. If $D$ is a subspace in the dual $X^*$ and

$$\langle Lx - b, a \rangle = 0 \text{ for all } a \in D,$$

such vector $x$ is sometimes called a weak solution of the equation. In case the subset $D$ is dense we have a real solution.

**Example.** Let us look the Poisson PDE

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = b(x, y)$$
defined on an open set \( \Omega \subset \mathbb{R}^2 \). Let us consider the space

\[
C_0^\infty(\Omega) = \{ \phi(x, y) : \phi \text{ is smooth and has compact support } \subset \Omega \},
\]

often called test functions.

The derivation of the weak solutions of the equation is now done as follows.

\[
\int \int_\Omega \left[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - b(x, y) \right] \phi(x, y) dxdy = 0.
\]

Applying Green’s theorem this can be written

\[
\int \int_\Omega \left[ \frac{\partial u}{\partial x} \frac{\partial \phi}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial \phi}{\partial y} + b(x, y) \phi(x, y) \right] dxdy = 0, \text{ or in shorter notation}
\]

\[
\int \int_\Omega \left[ u_x \phi_x + u_y \phi_y + b \phi \right] dxdy = 0.
\]

Solutions of this equation are called weak solutions of the Poisson equation. Such a technique is central in the Galerkin method for finite element solutions for PDE:s.

### 8.8 Some classes of operators

Some special types of linear operators appear in various contexts. An operator \( A : X \rightarrow Y \) is called **finite dimensional** if the image \( A(X) \subset Y \) is finite dimensional. The operator is called **normal** if \( A^* A = A A^* \). The operator is called **unitary** if \( A^* A = I \). Operator is **nilpotent** in case \( A^k = 0 \) for some power \( k \). An operator is **compact** if the image of the unit ball \( \{ x : \|x\| = 1 \} \) is contained in a compact set.

**Example.** Let \( X = C[0, 1] \) and \( I \) be a set of interpolation points \( z_1, z_2, \ldots, z_k \in [0, 1] \). Define an operator

\[
T : C[0, 1] \rightarrow P^k[0, 1] \subset C^\infty[0, 1]
\]

which maps a function \( f \) into the interpolation polynomial

\[
Tf = p_I(f).
\]

This operator is clearly finite dimensional, \( \text{dim}T(X) = k \).
Compact operators are very close to finite dimensional operators and this determines their behavior. The following is true. An operator

$$T : X \to Y$$

is compact if and only if there are a sequence of finite dimensional operators $$T_n : X \to Y$$ such that

$$\|T_n - T\| \to 0.$$  

**Example** Consider the integral operator

$$(Tu)(s) = \int_a^b k(s,t)u(t)dt$$

on the space of $$L^1[a, b]$$ for instance. Assume that the kernel function $$k(s,t)$$ can be approximated by a sequence of simple functions

$$k_n(s,t) = \sum k\chi_{A(k)}(s,t).$$

Define operator $$T_n$$ as follows

$$(T_n u)(s) = \int_a^b k_n(s,t)u(t)dt.$$  

It is clear that these are finite dimensional operators and

$$\|T - T_n\| \to 0.$$ 

So this generic integral operator formula, with some mild assumptions on $$k(s,t)$$, generates a compact operator.

The possibility to approximate a compact operator by finite dimensional operators is used in many contexts. One can show for instance that the range $$T(x)$$ of a compact operator must always be separable, the adjoint operator $$T^*$$ is also compact etc. An interesting example in Hilbert space is the operator

$$Ax = \sum \lambda_i \langle x, e_i \rangle e_i,$$

where $$e_i$$ is a sequence of orthonormal vectors. One can show that this operator is compact if and only if $$\lambda_i \to 0$$. All compact operators in a Hilbert space have such a representation.
8.9 Eigenvalues

Recall that for a matrix $A \in \mathbb{R}^{n \times n}$ an eigenvalue $\lambda$ is a number which solves the equation $Ax = \lambda x$ or $(A - \lambda I)x = 0$ for some vector $x \neq 0$. This is equivalent to saying that the linear operator $(A - \lambda I)$ has nontrivial null-space or $(A - \lambda I)$ is non-invertible.

The same idea can be transferred to arbitrary normed space $X$. If $T: X \to X$ is a linear operator on $X$, then $\lambda$ is an eigenvalue of $T \iff Tx = \lambda x$. This means also that the linear operator $T - \lambda I$ does not have an inverse operator. In normed spaces one usually distinguishes the cases

$$T - \lambda I \quad \text{has an inverse operator}$$
$$T - \lambda I \quad \text{has an inverse operator and it is bounded.}$$

The set

$$\{ \lambda : T - \lambda I \quad \text{does not have a bounded inverse operator}\}$$

is called the spectrum of the operator $T$.

Study of eigenvectors and eigenvalues of linear operators is called spectral theory. Eigenvalues and eigenvectors appear in many contexts, including spectral methods in solving PDE:s, integral equations etc.

**Example:** Integral equation $\lambda u(t) - \int_a^b k(t,s)u(s)ds = v(t)$ can be written as $\lambda u - Tu = v$, where $Tu = \int_a^b k(t,s)u(s)ds$. Here we see the familiar structure $(\lambda I - T)u = v$.

**Example:** This equation represents a model of forced vibration

$$u''(t) + \lambda u(t) = v(t) \quad \text{with } u(0) = u(1) = 0.$$  

Using the notation with differential operator $D$ this reads

$$(D^2 + \lambda I)u = v.$$  

Eigenvalues in this case would reveal the typical resonant frequencies (eigenfrequency) and the eigenvectors would be the corresponding function shapes $u_k(t)$, often called vibration-eigenmodes in engineering.

**Example:** The following is an equation of a vibrating membrane

$$\lambda u - \nabla^2 u = f, \quad \text{where } \Omega \subset \mathbb{R}^2, \text{ and } u = 0 \text{ on } \partial \Omega.$$
This can be written \((\lambda I - \nabla^2)u = f\).

This model has been used in medical context to study the dynamics of heart valve with an intention to diagnose flawed structure or malfunction from the measured electrocardiogram.

**Example:** The following is a simple heat equation in 1D.

\[ u_t - ku_{xx} = 0. \]

Boundary conditions are given as

\[ u(-1,t) = 0, \quad u(1,t) = 0, \quad u(x,0) = f(x) \]

The usual method of separating variables will illustrate the idea of eigenvector method in solving PDE:s. Let us look for solutions of the following type \( u(x,t) = M(x) \cdot L(t) \). By differentiating this expression (twice on \( x \)) and separating \( x \) and \( t \) on different sides of the equation we will get

\[ \frac{L'(t)}{L(t)} = k \frac{M''(x)}{M(x)} = \lambda \]

The last equal sign is because we can conclude that left and right sides are no longer functions of either \( x \) or \( t \). This equation will split into two separate ODE:s

\[ kM''(x) = -\lambda M(x) \]

\[ L'(t) = \lambda L(t) \]

The equation on \( M(x) \) has a characteristic equation \(-k \alpha^2 = \lambda\) with roots \( \alpha = \pm i \frac{\sqrt{\lambda}}{\sqrt{k}} \) so the solutions are

\[ M(x) = A \cos \left( \frac{\sqrt{\lambda}}{\sqrt{k}} x \right) + B \sin \left( \frac{\sqrt{\lambda}}{\sqrt{k}} x \right). \]

The boundary conditions require that \( M(-1) = M(1) = 0 \) and so only certain values for \( \lambda \) satisfy this. We have

\[ \lambda_n = k \left( \frac{n\pi}{2} \right)^2 \quad n = 0, 1, \ldots. \]

These acceptable values of \( \lambda \) are the eigenvalues of this problem and the corresponding eigenmodes are

\[ M_n(x) = A \cos \left( \frac{n\pi x}{2} \right) + B \sin \left( \frac{n\pi x}{2} \right) \]

When the found values of \( \lambda_n \) are inserted to the \( L \)-equation we obtain the solutions \( L_n(t) \). The solution of the original problem is finally sought as series representation

\[ u(x,t) = \sum_{n=1}^{\infty} \alpha_n M_n(x) L_n(t) \]
9 Operators in Hilbert Space

In a Hilbert space $H$ the formula $x \rightarrow \langle x, a \rangle$ defines a linear functional for each $a \in H$. A very important Riez theorem states that these are the only linear functionals in $H$ and the norm of this functional is $\|a\|$. This means that the dual space of $H$ is identical with $H$, that is $H = H^*$. As an example $(l^2)^* = l^2$.

9.1 Self Adjoint Operator

If $T : H \rightarrow H$ is an operator on $H$, the **adjoint operator** $T^* : H \rightarrow H$ is defined by the formula

$$\langle T^*x, y \rangle = \langle x, Ty \rangle$$

If $V$ and $W$ are any normed spaces and $T : V \rightarrow W$ is a linear operator the above formula defines an adjoint operator $T : V^* \rightarrow W^*$ between the corresponding dual spaces. However in Hilbert space $H = H^*$ so the dual operator is acting on the space itself.

In the case of $\mathbb{R}^n$ this reduces to familiar notion. The inner product is now $\langle x, y \rangle = x^T y$. If $H = \mathbb{R}^n$ and $A \in \mathbb{R}^{n \times n}$ the adjoint of the operator $x \rightarrow Ax$ will be simply $x \rightarrow A^*x$, that is taking the adjoint equals taking transpose.

The following condition defines a **self adjoint operator**

$$\langle Tx, y \rangle = \langle x, Ty \rangle.$$ 

This means that $T = T^*$. In the case of a matrix $A \in \mathbb{R}^n$ this condition means a symmetric matrix as the following calculation shows

$$\langle Ax, y \rangle = \langle x, Ay \rangle, \quad \forall x, y$$

$$(Ax)^T y = x^T (Ay),$$

$$x^T A^T y = x^T Ay$$

$$A^T = A$$

Self adjoint operators have many useful properties. The eigenvectors corresponding to distinct eigenvalues are always orthogonal. Assume $T : H \rightarrow H$ is self adjoint, $\lambda \neq \mu$ and

$$Tu = \lambda u, \quad Tv = \mu v$$
The we can calculate

\[ \lambda \langle u, v \rangle = \langle \lambda u, v \rangle = \langle Tu, v \rangle = \langle u, Tv \rangle = \langle u, \mu v \rangle = \mu \langle u, v \rangle \]

Since \( \lambda \neq \mu \) we must have \( \langle u, v \rangle = 0 \) which means that \( u \perp v \).

### 9.2 Example

To find the adjoint of a given operator requires careful computing. As an example think an operator \( A : L^2(\mathbb{R}) \to L^2(\mathbb{R}) \) defined as

\[ u(t) \to a(t)u(t + 1), \]

where \( a(t) \) is a bounded integrable function. Find the adjoint operator \( A^* \). Is the operator \( A \) self adjoint?

### 9.3 Spectral representation of an operator

Let \( \{u_n\} \) be an orthonormal sequence \( \in H \) and

\( \{\alpha_n\} \) a square summable sequence of scalars \( \sum_1^{\infty} |\alpha_i|^2 < \infty \).

Define an operator \( T : H \to H \) by formula

\[ Tx = \sum_1^{\infty} \alpha_i \langle x, u_i \rangle u_i \]

Then \( Tu_k = \sum_1^{\infty} \alpha_i \langle u_k, u_i \rangle u_i = \alpha_k u_k \) so each \( u_k \) is an eigenvector and \( \alpha_k \) is an eigenvalue. Moreover the following calculation shows that the operator is self adjoint.

\[ \langle Tx, y \rangle = \left( \sum_1^{\infty} \alpha_i \langle x, u_i \rangle u_i, y \right) \]

\[ = \sum_1^{\infty} \alpha_i \langle x, u_i \rangle \langle u_i, y \rangle, \text{ and similarly} \]

\[ \langle x, Ty \rangle = \sum_1^{\infty} \alpha_i \langle y, u_i \rangle \langle u_i, x \rangle \]
Let us study a case where \(T : H \to H\) is a self adjoint operator with finitely many eigenvalues \(\{\lambda_1, \lambda_2, \ldots, \lambda_n\}\) eigenvalues. For each \(\lambda\) we can define a set
\[
H(\lambda) = \{x \in H \mid Tx = \lambda x\}.
\]
This is always a subspace and it is called an **eigenspace**. It is clear that \(\dim H(\lambda) > 0 \iff \lambda = \lambda_i\) for some \(i\).
Due to a remark above \(H(\lambda_i) \perp H(\lambda_j)\) whenever \(\lambda_i \neq \lambda_j\).

Let us assume, that the eigenspaces span the whole space (it happen when the operator has enough eigenvectors). Then we can write
\[
H = H(\lambda_1) \oplus H(\lambda_2) \oplus \ldots \oplus H(\lambda_n).
\]

Let \(P_i : H \to H(\lambda_i)\) be the natural orthogonal projection on the eigenspace \(H(\lambda)\). Then for each \(x\) we can derive the following decomposition
\[
\begin{align*}
x &= P_1x + P_2x + \ldots + P_nx \\
Tx &= T(P_1x + P_2x + \ldots + P_nx) \\
Tx &= T(P_1x) + T(P_2x) + \ldots + T(P_nx) \\
Tx &= \lambda_1 P_1x + \lambda_2 P_2x + \ldots + \lambda_n P_nx \quad \text{which means that} \\
T &= \lambda_1 P_1 + \lambda_2 P_2 + \ldots + \lambda_n P_n
\end{align*}
\]
The operator has been decomposed into a combination of projections.

### 9.4 Spectral decomposition on matrices

In the finite dimensional case we have \(H = \mathbb{R}^n\) and a symmetric matrix \(A \in \mathbb{R}^{n \times n}\) \(A^\top = A\). This matrix has orthonormal eigenvectors \(\{u_1, u_2, \ldots, u_n\}\). Let the eigenvalues be \(\{\lambda_1, \lambda_2, \ldots, \lambda_n\}\).

Let us form a matrix of the eigenvectors \(P = [u_1, u_2, \ldots, u_n]\). Due to well known facts in theory of matrices this \(P\) is now orthogonal \(P^\top P = I\) (why?) and \(A = PDP^\top\) where \(D = [\lambda_1 \ldots \lambda_n]\) is the diagonal matrix of eigenvalues.

Columnwise matrix multiplication gives
\[
A = PDP^\top = [u_1, u_2, \ldots, u_n]D[u_1, u_2, \ldots, u_n]^\top
\]
one gets
\[
A = \lambda_1 u_1u_1^\top + \lambda_2 u_2u_2^\top + \ldots + \lambda_n u_nu_n^\top
\]
This is the spectral decomposition of a symmetric matrix.
10 Fourier Analysis

The classical Fourier-analysis is an example of orthonormal decomposition, basis functions etc. Let us consider the space $H = L_2(\mathbb{R})$.

The functions $e_k = \{e^{ikt}\}$ form an orthogonal system in this space. One can show that

$$\langle e_k, e_m \rangle = \int_{-\pi}^{\pi} e^{ikt} e^{-imt} dt = 0 \text{ for all } k \neq m.$$ 

Given a function $f(t)$ defined on the interval $[-\pi, \pi]$ we want to represent the function using the orthonormal basis

$$f = \sum_k \langle f, e_k \rangle e_k = \sum_k \alpha_k e^{ikt}$$

If such decomposition exist, we know that due to orthogonality

$$\alpha_k = \langle f, e_k \rangle = \int_{-\pi}^{\pi} f(t) e^{-ikt} dt$$

In case the function $f(t)$ does not admit the above decomposition, we know that the series anyhow gives the best approximation, the orthogonal projection of the function $f(t)$ on the subspace spanned by the functions $e^{ikt}$.

10.1 Fourier Transform

Let $f(t)$ be a function defined on the whole real axis $-\infty, \infty$. We generate the Fourier series representation of $f(t)$ on an interval $-T/2, T/2$. Define $\omega = 2\pi/T = \omega(T)$. The functions $\{e^{in\omega t}\}$ are now an orthonormal basis in the Hilbert space $L^2[-T/2, T/2]$ and we can generate the Fourier decomposition

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega t} = \sum_{n=-\infty}^{\infty} c_n(T) e^{in\omega t}$$

The coefficients are

$$c_n(T) = \omega \frac{\omega}{2\pi} \int_{-T/2}^{T/2} f(t) e^{-in\omega t} dt$$
This can be written as

\[ f(t) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{i\omega t} G(i\omega)\omega, \]

where

\[ G(i\omega) = \frac{T/2}{-T/2} \int f(t)e^{-i\omega t} dt. \]

When \( T \to \infty \), this series will approach the integral

\[ f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} F(\omega) d\omega, \]

where \( F(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} f(t) dt \) is called the Fourier transform of \( f(t) \). The preceding integral above defines the Inverse Fourier Transform \( F(\omega) \to f(t) \). The Fourier transform of \( f \) is often denoted also by the symbol \( \hat{f} \).

### 11 Time/frequency localization, Wavelets

The Fourier transform is meant to reveal, how a function is composed as a mixture of frequencies. In the case of Fourier series on a finite interval one obtains a denumerable decomposition of harmonic components. In the case of Fourier transform we have a continuum of harmonics mingled together and the function is assumed to be defined on the whole real axis.

One would often like to analyse the frequency composition of a function that changes its behavior in time. We may have a signal which exhibits local variations in time. Think a voice/audio signal with transient vibrations, seismic signals, syllables in speech, chirps in natural sounds etc.

Fourier transform is not able to detect local passing structures in the signal, like localise the time when a sound is uttered. Fourier analysis is about the global behavior of the function. Some examples are illustrated below. The Fourier transform of a monochromatic siren wave function is concentrated at one point, the single frequency \( \omega \). On the other hand if one has an infinitely sharp pulse (so called delta function) at time point \( t_0 \) the Fourier transform will be constant over the entire spectrum \( \omega \in (-\infty, \infty) \). If we have a rectangular pulse on the interval \(-T, T\), then Fourier transform does not reveal in any way, where this pulse happened. Any translate of the pulse would give the same Fourier transform.
To analyse time-localised or similarly in image analysis spatially localised features in a signal, one solution is windowed Fourier transform. The idea is to cut a piece of the signal multiplying it by a local function $\phi(t - \tau)$ and computing the Fourier transform of the windowed function

$$f \rightarrow \phi f \rightarrow F[\phi f] \quad (2)$$

When the window is moved around by shifting it $\phi(t - \tau)$ one can analyze the local frequency content everywhere. One possible window function is the rectangle function. However, the sharp edges will cause embarrassing ripples in the transform. To overcome this a smoother window function has been proposed. A famous one is

$$\phi(t) = \frac{1}{2\pi\alpha} e^{t^2/4\alpha}$$
This is the well known Gaussian function and is called **Gabor Filter**. The windowed Fourier transform thus generated is called **Gabor transform**.

### 11.2 Continuous wavelet transform CWT

We study next a linear filter called continuous wavelet transform. The basis is a function \( \psi \in L^2(\mathbb{R}) \) which satisfies the condition

\[
\int_{-\infty}^{a} \psi dt = 0
\]

The function \( \psi \) will be called "Mother-wavelet". The specific properties of this function will appear later.

The function \( \psi \) generates a transformation \( W: L^2(\mathbb{R}) \to L^2(\mathbb{R}^2) \), mapping \( f(t) \to W(a,b) \) where

\[
W(a,b) = \int_{-\infty}^{\infty} f(t) \cdot \frac{1}{\sqrt{a}} \psi \left( \frac{t - b}{a} \right) dt = \int_{-\infty}^{\infty} f(t) \cdot \psi_{a,b}(t) dt = \langle f, \psi_{a,b} \rangle
\]

The notation

\[
\psi_{a,b}(t) = \frac{1}{\sqrt{a}} \psi \left( \frac{t - b}{a} \right)
\]

simply means the translated and dilated versions of the base function (mother) \( \psi \). The coefficient in the front means a normalizing constant giving

\[
\int_{-\infty}^{\infty} |\psi_{a,b}|^2 dt = \int_{-\infty}^{\infty} |\psi|^2 dt = 1
\]

Note the familiar inner product

\[
W(a,b) = \langle f, \psi_{a,b} \rangle = f(b) \cdot \psi_{a,0}(-b).
\]

Note also that the CWT transforms the signal \( f(t) \) (one variable function) into an image which is a two-variable function \( W(a,b) \).

Possible mother wavelets are so called Haar wavelet function

\[
H(t) = \chi_{[0,\frac{1}{2}]} - \chi_{[\frac{1}{2},1]}
\]

or Mexican hat which has the form of the function

\[
(1 - t^2/\sigma^2) \exp \left[ -t^2/2\sigma^2 \right].
\]
Morlet wavelet has the shape
\[ e^{-t^2} \cos(\alpha t), \]
where the constant is \( \alpha = \pi \sqrt{2/\ln 2} \).

**Example:** The following example will describe the nature of the CWT. Select as the mother-wavelet for instance Haar wavelet or Mexican Hat. Define
\[ f_1(t) = \sin \left( \frac{\pi t}{100} \right), \quad \text{if} \quad 0 \leq t \leq 10; \]
\[ f_2(t) = \sin \left( \frac{\pi t}{200} \right), \quad \text{if} \quad 5 \leq t \leq 15, \]
and set \( f = f_1 + f_2 \). The computation creates a two variable function \( W(a, b) \) depicted in the following image.

### 11.3 Redundancy of CWT

Many features of Fourier transform are also true for CWT. The transform
\[ f(t) \leftrightarrow W(a, \tau) \]
is reciprocal. One can reconstruct the function \( f(t) \) from the Wavelet transform \( W(a, \tau) \) by an inverse transform. For the formula \( f \leftarrow W \) see[]. Important feature is that the transform \( W(a, b) \) contains a lot of redundant information. One can reconstruct the function \( f(t) \) knowing the value of the CWT in only at a denumerable discrete set of points.

This means that instead of all dilatations and transformations
\[ \psi \left( \frac{t - \tau}{a} \right) \]
we only need to use integer translates \( \tau = l \) and the dyadic dilatations \( a = 2^k \). In this way we arrive to the discrete WT.

### 11.4 Discrete wavelet transform DWT

The discrete version of wavelet transform is now
\[ f(t) = \sum_k \sum_l d(k, t) \cdot 2^{-k/2} \cdot \psi \left( 2^{-k} t - l \right) \]

The wavelet transform - when successully computed - means decomposing the signal into components that will dissect the signal into pieces of information. The dilatated versions
\[ \psi \left( 2^{-k} t \right), \quad -\infty < t < \infty \text{ and } k \in \mathbb{Z} \]

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will be able to feel or diagnose structures of the signal at different scales. The translates represent the localization of the events (in time or space).

The discovery and choice of the mother wavelet function $\psi(t)$ will in general be the crucial exciting part of the wavelet theory. The powerful properties depend on the choice.

### 11.5 Haar MRA

A simple example of a wavelet is Haar wavelet $H(t) = \chi_{[0, \frac{1}{2}]} - \chi_{[\frac{1}{2}, 1]}$

This is conceptually clear but not very useful in applications. Due to the simplicity it can be used to explain the main ideas.

The purpose of the choice of the mother wavelet analysis is to generate a splitting of the signal into approximations which represent the information of the signal on different scales. Such decomposition is called **Multiresolution Analysis MRA**.

Simplest example is the following, called Haar MRA. We study a function $f(t)$ defined on the real axis. We define a set of functions $f_k(t)$ which are constants ($= \text{average of } f$) on the dyadic intervals $[2^k l, 2^k (l + 1)]$

\[
\begin{align*}
  f_0(t) &= \frac{1}{l+1} \int_{l}^{l+1} f(\tau) d\tau, & \text{if } l \leq t < l + 1; \\
  f_1(t) &= \frac{2}{2l+2} \int_{2l}^{2l+2} f(\tau) d\tau, & \text{if } 2l \leq t < 2l + 2; \\
  f_{-1}(t) &= \frac{1}{2^l} \int_{\frac{l}{2}}^{\frac{l+1}{2}} f(\tau) d\tau, & \text{if } \frac{l}{2} \leq t < \frac{l}{2} + \frac{1}{2}.
\end{align*}
\]

Define a set of functions

\[ V_k = \{ f \mid f \equiv \text{constant on } 2^k l \leq t < 2^k (l + 1) \}. \]

This is a subspace in the Hilbers space $L^2$, or $V_k \subset L^2$. In fact we have generated a chain of nested subspaces

\[ \ldots \subset V_2 \subset V_1 \subset V_0 \subset V_{-1} \subset \ldots \subset V_{-k} \subset L^2(\mathbb{R}) \]

It is natural to define

\[ V_\infty = \{ 0 \} \text{ and } V_{-\infty} = L^2(\mathbb{R}) \]
12 Calculus in Banach Space

Next we discuss integrating of functions with values in a Banach space $X$. Let $(\Omega, \Sigma, \mu)$ be a measure space and $u : \Omega \to X$ a measurable function.

**Example:** We have corrosive messy liquid in a tube with a profile of acidity varying in $x$ and $t$. Let $\Omega = [0, T]$ be the time period of the experiment. Let $E = L^1[0, 1]$ be the space of acid concentration profiles in the tube situated at $x \in [0, 1]$. The acididy profile $f(x, t)$ in changing in $x-$direction and pulsating randomly in time $t$. Here $(\Sigma, \mu)$ models the probability structure regarding how often and how long an acidity profile $f(x, t)$ persists when randomly hopping in $E = L^1[0, 1]$.

In this way we have defined a function $u : [0, T] \to L^1[0, 1]$ which is mapping $t \to u(x, t)$. In the following we explain what we mean by the integral

$$\int_\Omega u d\mu = \int_0^T u d\mu,$$

which is supposed to represent the accumulated total corrosive effect on the pipe along the distance $x$.

12.1 Bochner integral

The following function $u : \Omega \to X$ is called a simple function

$$u(\omega) = \sum_{i=1}^n u_i \chi_{E_i}(\omega).$$

Here the sets $E_i \subset \Omega$ are disjoint measurable sets.

We define the integral of the function $u$ as follows

$$\int_\Omega u d\mu = \sum_{i=1}^n u_i \mu(E_i).$$

For an arbitrary measurable subset $A \subset \Omega$ the integral is defined

$$\int_A d\mu = \sum_{i=1}^n u_i \mu(E_i \cap A).$$

Let $u : \Omega \to X$ be an arbitrary function on $\Omega$ and $A \subset \Omega$. We assume that there is a sequence of approximating simple functions $u_n : \Omega \to X$ such that
\[ \|u_n(\omega) - u(\omega)\| \to 0, \text{ for all } \omega. \]

Now if \( \int_{\Omega} \|u\| \, d\mu < \infty \), the integrals of the simple functions \( \{ \int_{E} u_n d\mu \} \)
will then become a Cauchy sequence in \( X \). The limit is called the Bochner integral of \( u(\omega) \)
\[ \int_{E} u d\mu = \lim_n \int_{E} u_n d\mu. \]

### 12.2 Gateaux derivative

Let \( X \) be a vector space \( Y \) a normed space and \( u: X \to Y, Y \) a function. If \( x \in X \) and \( \eta \in Y \), we define
\[ Du(x) \cdot \eta = \lim_{\alpha \to 0} \frac{u(x + \alpha \eta) - u(x)}{\alpha}. \]
If this limit exists, it is called the directional derivative of \( u \) at point \( x \) to direction \( \eta \). The directional derivative is a function \( Du(x): X \to Y \) (defined maybe only on some directions).

The following fundamental fact has important applications in optimization, especially optimal control and optimal shape design.

**Theorem 12.1.** Assume that \( f \) is a function \( f: X \to \mathbb{R}, x \) is an extremal point of this function and the directional derivative \( Df(x) \) exist, then \( Df(x) = 0 \)

Assume that we need to find a function shape \( y(t) \) so that it minimizes a given utility/cost functional. The cost functional may represent costs of manufacturing, use of raw material, penalty due to technical performance etc. In optimal control we may be interested in optimizing cash flow, minimizing time to finish etc.

**Example:** We need to find a function shape \( y(t) \) on the interval \([0,1]\) so that it minimizes the functional \( F(y) \) given below.

\[ F: C[0,1] \to \mathbb{R}, \quad F(y) = \int_0^1 ty(t) + y(t)^2 \, dt \]

We compute the Gateaux derivative of this functional at point \( y(t) \in C[0,1] \) into direction \( \eta = \eta(t) \in C[0,1] \).
\[ F(y + \alpha \eta) - F(y) = F(y(t) + \alpha \eta(t)) - F(y(t)) \]
\[ = \int_0^1 t [y(t) + \alpha \eta(t)] + [y(t) + \alpha \eta(t)]^2 dt - \int_0^1 ty(t) + y(t)^2 dt \]

From this one can easily derive the limit (Exercise!)
\[ DF(y)\eta = \int_0^1 (t + 2y(t))\eta(t)dt \]

If there is a minimum for the functional \( F(y(t)) \) it should satisfy
\[ DF(y)\eta = 0 \quad \forall \eta \]

Especially by setting \( \eta = (t + 2y(t)) \) we get
\[ DF(y)\eta = \int_0^1 (t + 2y(t))^2 dt = 0 \]
This is possible only if \( t + 2y(t) = 0 \) so we have shown that the candidate for an optimal shape is
\[ y(t) = -\frac{1}{2} t \]

**Example:** As an exercise compute the Gateaux derivative of the following cost functional
\[ F(y(t)) = \int_0^1 x(t)^2 + x(t)x'(t) dt \]

Often the optimization should be carried out in a subspace of functions, expressed by boundary conditions, such as
\[ V = \{ x(t) \in C^1[0, l] : x(0) = x(1) = 0 \} , \text{or} \]
\[ W = \{ x(t) \in C^1[0, l] : x(0) = 0, x'(1) = 0 \} \quad \text{etc.} \]

### 12.3 Frechet derivative

There is another way to define the derivative of a function \( u(x) \) between normed spaces \( X \) and \( Y \). Assume that the function \( u \) can be locally approximated in the neighbourhood of point \( x \) by a linear function. More precisely if \( L \) is a linear operator \( L : X \to Y \) such that
\[ u(x + h) = u(x) + Lh + \|h\| \phi(h) \]

and \( \phi(h) \to 0 \) as \( h \to 0 \). Then we say that the linear operator \( L \) is the Frechet derivative of \( u \) at point \( x \). We denote this operator by \( L = du(x) \). If \( u \) has Frechet derivative at \( x \), it is easy to see that it also has Gateaux derivative for every direction.

**Example** In optimal control theory one often has a cost functional of the generic form
\[
F(x(t)) = \int_a^b u[x(t), x'(t), t] dt.
\]
Here \( u = u(x, y, z) \) is a function in \( C^2(\mathbb{R}^3) \) that is having second partial derivatives. One can show (CP page 98) that the Frechet derivative of this functional is
\[
\int_a^b \left[ \frac{\partial u}{\partial x} - \frac{d}{dt} \left( \frac{\partial u}{\partial y} \right) \right] hdt + \left[ \frac{\partial u}{\partial y} \right]_a.
\]
This is the basis of Euler-Lagrange equations in variational calculus.

### 13 Stochastic Calculus

#### 13.1 Random variables

A random experiment refers to a system with uncertain, unpredictable outcome \( \omega \). The sample space \( \Omega \) is the universe of all possible outcomes. Subsets of \( \Omega \) are called events. The events constitute a collection called \( \sigma \)-algebra \( \Sigma \). A random variable is a numeral variable that is determined by the outcome \( \omega \). The probability

Mathematically speaking probability \( p \) is now a measure on the \( \sigma \)-algebra \( \Sigma \). A random variable

\[ x : \Omega \to \mathbb{R} \]

is a measurable function on \( \Omega \) and we assume that \( x \in L^2(\Omega) \).

The expectation of \( x \) is defined as \( E(x) = \int_\Omega x dp \). If \( x, y : \Omega \to \mathbb{R} \) are two random variables, the covariance between them is defined as

\[
cov(x, y) = E(x - Ex)(y - Ey)
= Exy - Ex \cdot Ey
\]
and the variance is
\[
Var(x) = cov(x, x) = E(x - Ex)^2
= Ex^2 - (Ex)^2
\]
Note: If for a random variable we have $\mu = E(x) = 0$ then

$$\text{variance} = \text{Var}(x) = E(x^2) = \int_{\Omega} x^2 dp = \|x\|_2^2.$$  

For Gaussian variables, if $E(x) = E(y) = 0$ then

$$\rho(x, y) = 0 \iff E(xy) = 0 \iff x, y \text{ are independent.}$$

### 13.2 Stochastic process

This chapter is an introduction about how functional analytic concepts and methods are used in stochastic calculus, analysis of stochastic processes and stochastic differential equations.

A stochastic process is a sequence or continuous chain of random variables $X_t : \Omega \to \mathbb{R}$ where $t \in [0, T].$ Here $\Omega$ is probability space. Some example could be $x(k) =$"number of calls in GSM - network”

$y(k) =$"share value of a company $X$”

$z(k) =$"diameter of copper wire”

$v(k) =$"amount of insurance claims in week k received by a car insurance company”

Knowledge of the stochastic behaviour of the process can be given as the joint probability distribution of samples

$$\{x(t_1), x(t_2), \ldots, x(t_k)\}$$

For instance in the case of monitoring the quality of continuous manufacturing process (paper weight, copper wire diameter) one can assume that the variable $x(t)$ has a probability distribution $x(t) \sim n(\mu(t), \sigma^2)$.

If the variable exhibits internal coupling between time point $t$ and $t + k$, this would be visible in the correlation coefficient of the joint distribution of the random vector $x(t), x(t + k)$.

### 13.3 Gaussian stochastic process

Multinormal distribution is defined as

$$f(x) = C \exp -\frac{1}{2}(x - \mu)^T A(x - \mu).$$

Here $A$ is the inverse of the covariance matrix which gives the structure of internal dependencies of the random vector $\{x(t_1), x(t_2), \ldots, x(t_k)\}$. 

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A Gaussian stochastic process is one where the vector \( \{ x(t_1), \ldots, x(t_k) \} \) has a multinormal distribution. The stochastic process has its characteristic features, expectation

\[
E(X(t)) = \mu(t)
\]

and the covariance function

\[
r(s, t) = \text{cov}(X(t), X(s)) = E(X(t) \cdot X(s)) - \mu(t) \cdot \mu(s).
\]

### 13.4 Stochastic differential equation

A familiar differential equation of forced vibration of mass-spring system can be used to model the behavior of car tire suspension, where the effect of the non-smooth road is appearing as a forcing term affecting the system. This equation (when changed into a 1st degree system) has a typical form

\[
x'(t) + a(t)x(t) = F(t)
\]

The complicated irregular shape of the non-smooth road surface suggest to model the forcing term as a stochastic process \( F(t) = b(t)w'(t) \) where \( w'(t) \) means so called white noise and \( b(t) \) it’s amplitude. In this way we arrive at a simple example of a **stochastic differential equation**

\[
x'(t) + a(t)x(t) = b(t)w'(t)
\]

The analysis of such an equation is done by transforming it into a corresponding equation about integrals

\[
x(t) = x(0) + \int_0^t a(s)x(s) ds + \int_0^t b(s) dw(s).
\]

To understand and analyse such equations one needs to know the basics of a **stochastic integral**.

Let us have a stochastic process \( \omega(t) : \Omega \to \mathbb{R} \) or \( \mathbb{R}^k \) where \( t \in [0, T] \). If we have a measurable function \( B \in L_2[0, T] \) we want to define what we mean by the integral of \( B(t) \) with respect to the stochastic process \( \omega(t) \)

\[
\int_0^T B(t) d\omega(t) = ?
\]

The definition of teh integral will be based on the idea of Riemann-Stieltjes integral, by studying the convergence of partial sums

\[
\sum_j B(t_j)[w(t_{j+1}) - w(t_j)],
\]
when the partition \{t_0, t_1, \ldots, t_n\} is refined. Notice that since \{\omega(t) \mid t \in [0, T]\} is a random orbit of the process, the integral will also become a random variable. More exactly we will have a random variable

$$\int_0^T B(t) d\omega(t) : \Omega \to \mathbb{R}.$$  

In most cases the integrating process \omega(t) will be of a special type called a Wiener process.

### 13.5 Wiener process and White noise

White noise (discrete) process is defined as a sequence \(x_n\) of independent identically distributes random variables with \(E(x) = 0\) Hence the covariance function is

$$\text{cov}(x_n, x_m) = r(n, m) = \begin{cases} \sigma^2 & m = n \\ 0 & m \neq n \end{cases}$$

One can also have a white noise process in k dimensions, where

\[ x_n = (x_{n1}, x_{n2}, \ldots, x_{nk}) \in \mathbb{R}^k \]

and \(E(x_n) = 0 = (0, 0, \ldots, 0)\). Then

\[ R(m, n) = E(x_n x'_m) = \begin{bmatrix} \sigma^2 & 0 & \cdots & 0 \\ 0 & \sigma^2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \sigma^2 \\ 0 & 0 & \cdots & \sigma^2 \end{bmatrix} \]

One can also define a continuos time White noise process \(x(t)\) where \(x(t) \sim n(0, \sigma^2)\) and \(\text{cov}(x(t), x(s)) = 0\) when \(t \neq s\).

Let \(w(t), t \geq 0\) be a Gaussian stochastic process, that is \(w(t) \sim n(0, \sigma^2)\). It is called a **Wiener process** if its covariance function is

\[ r(s, t) = \text{cov}[w(s), w(t)] = \sigma^2 \min(s, t). \]

This assumption will imply the following

1. \[ E \left[ (w(t) - w(s))^2 \right] = \sigma^2(t - s) \]
2. If \([s_1, t_1], [s_2, t_2]\) are disjoint intervals, then \(w(s_1) - w(t_1), w(s_2) - w(t_2)\) are independent.

The uncertainty regarding the value of the variable increases linearly in time and the process has independent increments.
13.6 Stochastic integral, an introduction

The integral of a simple function

\[ B(t) = \sum_{i=1}^{n} B_i \chi_{E_i}(t) \]

is clearly

\[ \int_{0}^{T} B(t)dw(t) = \sum_{i=1}^{n} B_i[w(t_{i+1}) - w(t_i)] \]

Since \( w(t) \) is a Wiener process, we have \( E[w(t_i)] = 0 \) for all \( i \). Hence also

\[ E\left[ \int_{0}^{T} B(t)dw(t) \right] = 0. \]

Due to the assumptions of Wiener process (variance rule, independence of increments) we can compute the \( L^2 \)-norm

\[ \left\| \int_{0}^{T} B(t)dw(t) \right\|_2^2 = \text{Var}\{\sum_j B_j[w(t_{j+1}) - w(t_j)]\} \]

\[ = \sum_j B_j^2 \text{Var}[w(t_{j+1}) - w(t_j)] \]

\[ = \sigma^2 \sum_j [t_{j+1} - t_j] \]

\[ = \sigma^2 \|B\|_2^2 \]

For a general measurable function \( B(t) \) we take a sequence of simple functions \( B_n(t) \) so that

\[ \|B_n - B\|_2 \to 0. \]

The integral will then be defined as a limit

\[ \int_{0}^{T} B(t)dw(t) = \lim_{n \to \infty} \int_{0}^{T} B_n(t)dw(t). \]

The limit exists because the values of the integrals \( \int B_n dw(t) \) are a Cauchy sequence in \( L^2 \)

\[ \left\| \int_{0}^{T} B_n(t)dw(t) - \int_{0}^{T} B_m(t)dw(t) \right\|_2 = \left\| \int_{0}^{T} B_n(t) - B_m(t)dw(t) \right\|_2 \]

\[ = \sigma \|B_n - B_m\|_2 \].
13.7 Black Scholes model

A famous example of a stochastic differential equation if the Black-Scholes model

\[ dS_t = \mu S_t dt + \sigma S_t dw(t) \quad \text{or equivalently} \]

\[ \frac{dS_t}{S_t} = \mu dt + \sigma dw(t). \]

This model is used to describe the time evolution of stock prices. The model has been the basis of financial trading since it can be used to compute option prices. The coefficient \( \mu \) is the natural interest rate, \( \sigma \) so called market-volatility and \( w(t) \) a Brownian motion describing the non-predictable impulses affecting the market.

13.8 System with noisy input

Let us consider a simple system modelled by a differential equation

\[ x'(t) = -ax(t). \]

The model may describe growth/decay and also a system with a tendency to approach an equilibrium \( x = 0 \). Assume that the system is also subject to outside perturbation, random impulses. We write

\[ x'(t) = -ax(t) + w'(t), \]

where \( w'(t) = \) white noise process, the derivative of a Wiener process \( w(t) \). Multiplying by \( \exp(at) \) we get

\[ e^{at}x'(t) + e^{at}ax(t) = e^{at}w'(t) \]

\[ \frac{d}{dt} \left[ e^{at}x(t) \right] = e^{at}w'(t) \]

Integrating this equation gives

\[ e^{at}x(t) = x(0) + \int_0^t e^{as}w'(s)ds \]

\[ x(t) = e^{-at}x(0) + \int_0^t e^{a(s-t)}w'(s)ds \]

\[ x(t) = e^{-at}x(0) + \int_0^t e^{a(s-t)}dw(t) \]
It is easy to generate numerical simulations of this model. The solution are - as they should be - random realizations of a stochastic process. Due to the properties of Wiener process we know that the increments
\[ \Delta w(t) = w(t + h) - w(t) \]
are Gaussian random variables with variance \( h\sigma^2 \) and so we have
\[ w(t + h) = w(t) + Z\sigma \sqrt{h}, \]
where \( Z \) is a random variable sampled from the standard normal distribution \( Z \sim n(0, 1) \). Using this one can generate simulated paths for \( x(t) \).

14 Optimization and introduction to optimal control

Vector space concept, especially Hilbert space methods are important tools in optimization theory. Let us start by some notes on bilinear forms. Let \( H \) be a Hilbert space and \( F = F(x, y) \) a function: \( H \times H \rightarrow \mathbb{R} \) that is linear with respect to both variables. Such functions are called bilinear forms.

A simple example in \( \mathbb{R}^3 \) would be
\[ F(x, y) = \sum_i a_{ij} x_i y_j = 4x_1 y_1 + 6x_1 y_2 - 2x_2 y_2 + x_2 y_3 \]

If \( L_1 \) and \( L_2 \) are linear functionals on \( H \) then
\[ F(x, y) = L_1(x) L_2(y) \]
is also a bilinear form.

A bilinear form is coercive if for some \( c > 0 \)
\[ F(x, x) \geq c \| x \|^2 \]
for all \( x \in H \).

Using wellknown theorems in Hilbert spaces and convexity argument one can prove the following (see Curtain Pritchard p 254).

**Theorem.** If \( F(x, y) \) is a continuous symmetric coercive bilinear form and \( L \) is continuous linear functional on \( H \), the functional
\[ J(x) = F(x, x) - L(x) \]
has a unique minimum point \( x^* \) and this point satisfies the equation
\[ F(x^*, y) = \frac{1}{2} L(y) \]
for all \( y \in H \).
14.1 Least squares minimization

Let $A : H_1 \to H_2$ be a bounded linear operator. We want to solve an equation

$$Ax = b.$$  

In many real situations we have an equation which does NOT have a solution. A simple example is an overdetermined system where errors in data make it not solvable. The idea of least squares solution is, instead of exact solution, minimize the functional $J(x) = \|Ax - b\|$. Now

$$J(x) = (Ax - b, Ax - b) = (A^*Ax, x) - 2(A^*b, x) + \|b\|^2 = F(x, x) - 2L(x) + \|b\|^2,$$

where $F$ is a symmetric continuous bilinear form anf $L$ a bounded linear functional. If now the operator $A^*A$ is coercive, or equivalently $A$ satisfies

$$\|Ah\|^2 \geq c\|h\|^2 \text{ for some } c > 0,$$

then we could use the previous theorem to find the LS-solution.

In this case we can also directly compute the Frechet differential of the functional $J(x)$. After a straightforward calculation one gets

$$dJ(x) = 2A^*Ax - 2A^*b.$$

The extremum point $u$ is found by solving the so called **normal equation**

$$A^*Au = A^*b.$$  

**Example** Consider finding a solution $u = u(x)$ on an interval $[a, b]$ for the differential equation arising for instance in mechanics

$$-\frac{d}{dx} \left[ a(x) \frac{du}{dx} \right] = f(x).$$

Here $a(x) > 0$ and the function $f(x)$ represents an outside loading/input onto the system. If for instance the loading is given as a step function with discontinuities, it may lead to an equation which does not allow an exact solution. Let us write

$$A = -\frac{d}{dx} \left[ a(x) \frac{d}{dx} \right].$$
One would like to apply the method explained above to find the least squares solution for $Au = f$ (Exercise). Note that the space $C^2[a, b]$ is not a Hilbert space. Let us consider instead the Sobolev space

$$H^2[a, b] = \{ u(x) \mid Du, D^2u \in L^2[a, b] \}.$$ 

Here $D$ and $D^2$ mean generalized derivatives (see Hall-Porshing p 108). This space will become a Hilbert space when we incorporate it with an inner product

$$\langle u, v \rangle = \int_a^b uv + DuDv + D^2uD^2vdx$$

and the norm

$$\|u\| = \int_a^b |u|^2 + |Du|^2 + |D^2u|^2 dx.$$ 

To carry out the task one needs to find the adjoint operator $A^*$ first.

### 14.2 Inverse problems and regularization

We have a linear operator $T : X \to Y$ between normed spaces. We are solving an operator equation $Tx = y$ in the case where this operator does not have a decent inverse operator. Either $T^{-1}$ is unbounded or the norm $\|T^{-1}\|$ is very large. In this case the problem is called ill-posed. In practice we cannot know $y$ exactly but our model is disturbed by some measurement error $\epsilon$, which means that we are actually solving an equation

$$Tx = y + \epsilon.$$ 

Small deviation in $y$ will now generate big or huge deviation in the solution $x = T^{-1}(y + \epsilon)$. This is the nature of so called inverse problems. An example of such phenomenon are the well-known integral operators $Tu(s) = \int k(s, t)u(t)dt$.

**Example.** In photography an accidental shift of the camera and perhaps poor focus will result in a ”blurred image”. If $u = u(x, y)$ is the correct image (colour density in gray scale) the blurred image $b = b(x, y)$ will be a result of an integral transform

$$b(x, y) = T[u(s, t)]$$

of the type described above, this time with double integral of course.
**Example.** Let us look the integral equation

\[
\int k(s, t)u(t)dt = y(s)
\]

We consider this integral operator \(T\) between normed spaces, where the norm of \(u(t) \in X\) is defined as

\[
\|u\| = \left[ \int |u(t)|^2 + |Du(t)|^2 dt \right]^{1/2}
\]

and as the norm of \(y(s) \in Y\) we take usual \(L^2\)-norm \(\|y\|_2\). Assume that we have measured an output \(\eta(s)\) which is known to be inaccurate, so some error is involved. Due to this error in the approximation \(\eta(s) \approx y(s)\) and the unbounded inverse of \(T^{-1}\) the solution of the integral equation can be far away from correct.

The following regularization idea tries to decrease the sensitivity of the solution. This is done by minimization

\[
\|Tu - \eta\|^2 + \alpha \|u\|^2 \to \text{minimum}!
\]

The idea is to control the error sum and at the same time keep the norm of the solution small. The parameter \(\alpha > 0\) is called regularization parameter. To achieve the minimum we compute the Frechet derivative of the functional

\[
J(u) = \|Tu - \eta\|^2 + \alpha \|u\|^2
\]

\[
= \langle Tu - \eta, Tu - \eta \rangle + \alpha \langle u, u \rangle
\]

\[
= \langle T^*Tu, u \rangle - 2 \langle T^*\eta, u \rangle + \langle \eta, \eta \rangle + \alpha \langle u, u \rangle
\]

We see that the Frechet derivative is

\[
dJ(u) = 2T^*Tu + 2\alpha u - 2T^*\eta.
\]

The minimum should satify teh equation

\[
T^*Tu + \alpha u = T^*\eta.
\]

From this we get

\[
u = (T^* + \alpha I)^{-1} T^*\eta.
\]
14.3 Example from earth science

Let \( p(x) \) be the permeability of soil at depth \( x \), which describes the ability of water to seep through the layer of earth. We denote by \( w(x, t) \) the amount of water (per unit volume) in the soil at depth \( x \) at time \( t \). In earth science one has derived a law which governs the permeation of water (after rain shower, let us say) into the soil. The equation modeling the process looks like this

\[
 w(x, t) = T(p(x)) = \int_{0}^{t} K(x, s)p(s)ds
\]

This equation means an inverse problem. The solution will give the “water hold-up” \( w(x, t) \).

15 Optimal Control

Let us study the following scheduling problem in the production of a biochemical substance in agrobio-industry. The substance is an important raw material in bakery and food production. The material is spoiled easily in storage, so the consumption of the material must be closely adjusted with the demand. The material is decaying with a small constant of decay \( \alpha = 0.04 \) (evaporation or some leakage). Adding a catalytic and nutritious material one can generate growth. More exactly let \( x(t) = \) the amount and \( u(t) = \) the amount of catalytic nutrient. The natural decay process is modeled by equation

\[
 x(t) = -\alpha x(t).
\]

The organic growth can be modelled by

\[
 x(t) = \beta x(t)u(t).
\]

The period for scheduling the production is \([0, T]\). The demand curve varies periodically as \( D(t) = 10 + sint(t) \).

The amount of the key product is described by the differential equation

\[
 x(t) = -\alpha x(t) + \beta x(t)u(t) \tag{1}
\]

We call the amount \( x(t) \) as the state of our system. This state can be controlled by adding the nutrient. The amount \( u(t) \) is called control variable of our system.
We want to make the production curve to be as close to the demand curve as possible. We try to minimize the total integral of \([x(t)D(t)]^2\). The second and simultaneous objective is to save the nutrient which is quite expensive to use. So we want to minimize also the square integral of \(u(t)\). So all together we want to find a schedule for applying \(u(t)\) and the resulting output \(x(t)\) following differential equation (*) so that the total cost function

\[
F(x(t), u(t)) = \int [x(t)D(t)]^2 + u(t)^2 dt
\]

is minimized. The optimal solution if we can find it is a function \(u*(t)\) in the space of functions \(C[0, T]\). Likewise the optimal solution regarding the production is also a function \(x*(t)\) in the space of functions \(C[0, T]\).

A generic form of optimal control problem can be formulated as follows. We have a system whose state at time \(t\) is described by \(x(t)\). The state usually is a vector \(x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]\), but in our case in above example we have 1-dimensional state vector. The system evolves in time and the time-path can be influenced by control function \(u(t)\) which also in general case may be vector valued.

The time-evolution of the system is determined by a differential equation (a system of DE:s in general)

\[
x(t) = f(x(t), u(t)), \quad x(0) = x_0
\]

We see, that the system has initial state \(x(0) = x_0\). The objective function that one wants to maximize (or minimize depending on application) has the form

\[
F(x(t), u(t)) = \int g(x(t), u(t))dt + \gamma(x(T))
\]

The latter term in this expression means that we may put a price on the final state of the system at the end \(x(T)\). It is called sometimes terminal payoff.

The task is to find function \(u = u^*(t)\) which satisfies equation (3) and maximizes functional \(F(x(t),u(t))\). The solution will be a pair \((x,u) = (x^*(t),u^*(t))\) giving the optimal control and the resulting optimal time evolution leading to maximum of the objective functional.

15.1 Examples

Drug therapy. Let \(x(t) =\) number of bacteria in intestinal system, \(u(t) =\) concentration of drug which can decrease the bacterial growth. Without
treatment the bacteria would be growing exponentially. Let us model the bacterial growth by

\[ x(t) = \alpha x(t) - u(t). \quad (5) \]

When initial infection is given \( x(0) = x_0 \), the wish is to minimize the amount of bacteria at the end of the treatment \( T \). At the same time taking a lot of drugs is also not nice, so we would like to minimize the negative effect. Let us model the harm caused by drug consumption by \( \int u(t)^2 dt \). Hence our objective would be:

\[ \text{minimize} \left\{ x(T) + \int u(t)^2 dt \right\}. \quad (6) \]

**Fishery model.** We model the total population of fish in a fishery farm by \( x(t) \). It may be a huge artificial tank or a natural lake. If the fish are allowed to breed and grow freely, the time evolution will follow a model

\[ x(t) = kx(t)(M - x(t)). \]

This model describes logistic growth towards a saturation limit value, in the case of fishery that maximum amount of fish that the volume can sustain, due to limitations of space and food. Let \( u(t) \) describe the intensity of fishing, more precisely the proportion of the total amount of fish that fishermen remove from the living space of the fish population. Hence the time evolution of the size of fish population is modeled by

\[ x(t) = kx(t)(M - x(t)) - u(t)x(t). \quad (7) \]

The commercial value of sold fish is modelled by a function

\[ p(x) = ax - bx^2 \]

with certain coefficients. This function will mean that the income does not grow linearly because the unit price will get lower when the supply of fish is increasing. A term diminishing returns is used. The cost of fishing is modeled by function \( cu(t) \). The net profit to be maximized is now

\[ F(x(t), u(t)) = \int e^{-\delta t}[ax(t)u(t) - bx(t)^2u(t)^2 - cu(t)]dt \quad (8) \]

Here we have added the discount factor which will emphasize the fact that income arriving early is more valuable because it can be invested to produce more profit.
15.2 Classical calculus of variations problem

Let us see as an example the problem of minimizing functional

\[ F(x(t)) = \int g(x(t), x(t), t) \, dt \quad (10) \]

over the class of differentiable functions on \([0,T]\) where the initial and final states are given \(x(0) = x_0\) and \(x(T) = x_T\). Here \(g = g(x, y, t)\) is a 3-variable function. By using rules of calculus, including chain rule and Taylor expansion, one can calculate the Gateaux derivative of the functional \(F(x(t))\). This means that we need to study the difference

\[ F(x + h) - F(x) = F(x(t) + h(t)) - F(x(t)) \]

\[ = \int g(x(t) + h(t), x(t) + h(t), t) - g(x(t), x(t), t) dt \]

\[ = \int \left[ \frac{\partial}{\partial x} g(x, x, t) h(t) + \frac{\partial}{\partial y} g(x, x, t) h(t) \right] dt + \text{remainder} \quad (11) \]

Because the function \(x(t) + h(t)\) must also satisfy the boundary conditions, we must have \(h(0) = h(T) = 0\). As \(h \to 0\), one can show that the remainder will vanish and so the first term will give the Gateaux derivative. Applying integration by parts and leaving out some arguments this will be

\[ dF(x)h = \int_0^T \left[ \frac{\partial g}{\partial x} - \frac{d}{dt} \frac{\partial g}{\partial y} \right] h(t) dt \quad (12) \]

Because \(h(0) = h(T) = 0\), the second term in the integration by parts will be zero, that is the substitution

\[ \left[ \frac{\partial u}{\partial y} h(t) \right]_0^T = 0. \]

If \(x = x(t)\) is an function that minimizes the functional, then the Gateaux derivative for all increment vectors \(h(t)\) should be = 0. This is possible only if

\[ \frac{\partial g}{\partial x} - \frac{d}{dt} \left( \frac{\partial g}{\partial y} \right) = 0 \quad (13) \]

By solving this so called Euler-Lagrange equation one gets the optimal function \(x(t)\).